Near vector spaces

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Declaration

I the undersigned hereby declare that the work contained in this thesis is my own original work and has not previously in its entirety or in part been submitted at any university for a degree.

Signature *Date* 5 January 1990

Summary

The preliminary material in Chapter 1 is included in order that this work be reasonably self-contained.

The main aim of this thesis is to give an exposition of the theory of near vector spaces, as introduced by J. André [1]. This is done in the second chapter. Concepts such as an F-group and its quasi-kernel are defined and investigated. A near vector space is then defined in terms of these notions, after which the theory of near vector spaces is developed.

Three examples are given in Chapter 3. Their quasi-kernels are investigated and, in each case, it is shown that the near vector space under consideration is not a vector space.

Finally, in the fourth chapter, certain known results concerning linear transformations of finite-dimensional vector spaces are recalled. These notions are then investigated for certain near linear transformations of a 2-dimensional near vector space.

Opsomming

Die inleidende eerste hoofstuk, in die besonder die afdeling oor afhanklikheidsrelasies, is hoofsaaklik vir verwysingsdoeleindes.

Die hoof doel van die tesis is om 'n uiteensetting van die teorie van byna-vektorruimtes, soos deur J. André [1] ontwikkel, te gee. Dit word in Hoofstuk 2 gedoen. Begrippe soos 'n F-groep en sy kwasi-kern word ondersoek. Byna-vektorruimtes word dan in terme hiervan gedefinieer. Hierna word die teorie van byna-vektorruimtes aangebied.

In Hoofstuk 3 word drie voorbeelde van byna-vektorruimtes gegee. Hulle kwasi-kerns word ondersoek en daar word, in elke geval, aangetoon dat die betrokke byna-vektorruimte nie 'n vektorruimte is nie.

Sekere bekende konsepte rakende lineêre transformasies van eindige vektorruimtes word in die vierde hoofstuk bespreek. Hierdie begrippe word dan ondersoek in die geval van sekere byna-lineêre transformasies van 'n 2-dimensionale byna-vektorruimte.

Preface

The main purpose of this thesis is to give an exposition of the theory of near vector spaces, as introduced by J. André [1]. This is done in Chapter 2.

In the third chapter three examples are given of near vector spaces which are not vector spaces.

The thesis concludes with an investigation, in Chapter 4, of certain near linear transformations of a 2-dimensional near vector space.

In order to make the presentation reasonably self-contained, some preliminary material is gathered together in the first chapter.

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Contents

Chapter 1

Introduction and preliminaries

1.1 Terminology

The terminology used is, except when stated to the contrary, that used by Meldrum [3].

1.2 Notes on vector spaces and generalizations thereof

A *(left) vector space* V over a division ring F is a set admitting addition (any two elements x, y of V possess a sum $x + y$ which is an element of V) and multiplication by scalars (for each x in V and for each λ in F, the product λx is defined and is an element of V) with the following properties:

(1) $x + y = y + x$.

(2) $x + (y + z) = (x + y) + z$.

(3) There is a zero element in V, denoted by 0, such that for all x in V, $0 + x = x$.

(4) For each x in V there is an element $-x$ in V such that $x + (-x) = 0$.

(5) $(\lambda\mu)x = \lambda(\mu x)$.

(6) $(\lambda + \mu)x = \lambda x + \mu x$.

(7) $\lambda(x + y) = \lambda x + \lambda y$.

(8) $1x = x$.

Note 1.2-1. (a) In a vector space V over a division ring $F, (V, +)$ is an Abelian group, while F can be regarded as a set of endomorphisms of V (for $\alpha \in F$, the endomorphism f_α of V is defined by $f_\alpha x := \alpha x$ for each $x \in V$).

(b) Let V be a vector space over a division ring F. Then, for every $\alpha, \beta \in F$ and for each $x \in V$, there is a $\gamma \in F$ (viz, $\gamma = \alpha + \beta$) such that $\alpha x + \beta x = \gamma x$.

(c) The sum of any two vectors of V which satisfy (b), also satisfies (b).

The observations in Notes 1.2-1(a), (b), respectively, lead to the concept of an F-group (Definition 2.1-1) and of its quasi-kernel (Definition 2.2-1). These notions in turn lead to the definition of a near vector space (Definition 2.4-1). The observation in Note 1.2-1(c) leads to the definition of a regular near vector space (Definition 2.4-15).

The concept of a near vector space was introduced and studied by André [1]. This theory is given in Chapter 2. Among others, are the decomposition theorem (Theorem 2.4-17) and the uniqueness theorem (Theorem 2.4-18) which state that there exists a unique direct decomposition of a near vector space into maximal regular subspaces. In Chapter 2, it is finally shown that a near vector space V which is not generated by a single element is a vector space if and only if V equals its quasi-kernel.

1.3 Dependence relations

The results of this section are used in Chapter 2, Section 3. Part of this theory is developed in [4],paragraph 10.

Let Q be a set and let 2^Q be the set of all subsets of Q. A relation between Q and 2^Q, denoted by $v \lhd M$, with $v \in Q$ and $M \subseteq Q$, is a *dependence relation* if the following three conditions are satisfied:

(D_1) $v \in M$ implies that $v \lhd M$.

(D_2) $w \lhd M$ and $v \lhd N$ for each $v \in M$, implies that $w \lhd N$.

(D_3) $v \lhd M$ and the falsehood of $v \lhd M \setminus \{u\}$ (denoted by $v \not\lhd M \setminus \{u\}$), imply that $u \lhd (M \setminus \{u\}) \cup \{v\}$.

Let $\lhd$ be a dependence relation on Q.

Theorem 1.3-1. *Let $M \subseteq N \subseteq Q$. If $w \lhd M$, then $w \lhd N$.*

Proof. If $v \in M$, then $v \in N$. Hence, by (D_1), $v \lhd N$. Therefore, by (D_2), $w \lhd N$.

Definition 1.3-2. (a) A finite subset E of Q is *independent* if there is no $v \in E$ such that $v \lhd E \setminus \{v\}$.

(b) An infinite subset M of Q is *independent* if every finite subset of M is independent.

Theorem 1.3-3. *Let $N \subseteq M \subseteq Q$. If M is independent, then N is independent.*

Proof. Let M be a finite set and suppose that N is not independent. Then there is a $v \in N$ such that $v \lhd N \setminus \{v\}$. But $N \subseteq M$. Hence $v \in M$ and, by Theorem 1.3-1, $v \lhd M \setminus \{v\}$. This contradicts the independence of M. Therefore N is independent. If M is an infinite set, the result follows from the definition.

Theorem 1.3-4. *Let $B \subseteq Q$ and $x \in Q$. If B is independent and $B \cup \{x\}$ is not independent, then $x \lhd B$.*

Proof. Since $B \cup \{x\}$ is not independent, there exists a finite subset B' of $B \cup \{x\}$ which is not independent. Hence there exists a $v \in B'$ such that $v \lhd B' \setminus \{v\}$. If $v = x$, then $B' \setminus \{v\} \subseteq B$. Hence, by Theorem 1.3-1, $x \lhd B$.

Suppose that $v \neq x$. Then, since $B' = B_e \cup \{x\}$ (with B_e a finite subset of B), $v \in B_e$. Therefore, since B_e is independent, $v \not\lhd B_e \setminus \{v\}$. Furthermore, $B' \setminus \{v\} = (B_e \cup \{x\}) \setminus \{v\} =: M$. Also, $M \setminus \{x\} = B_e \setminus \{v\}$. So $v \lhd M$ and $v \not\lhd M \setminus \{x\}$. Hence, by (D_3), $x \lhd (M \setminus \{x\}) \cup \{v\}$. But $(M \setminus \{x\}) \cup \{v\} = (B_e \setminus \{v\}) \cup \{v\} = B_e$. Hence $x \lhd B_e$. Therefore, by Theorem 1.3-1, $x \lhd B$.

Definition 1.3-5. Let M and N be subsets of Q. Then M *depends on* N (M is generated by N) if, for each $v \in M$, there exists a finite subset N' of N such that $v \lhd N'$.

Note 1.3-6. (a) Let M and N be subsets of Q. Suppose that M depends on N. Then, by Theorem 1.3-1, $v \lhd N$ for all $v \in M$.

(b) Let M be a subset of Q and N be a finite subset of Q. Then M depends on N if and only if $v \lhd N$ for each $v \in M$.

Theorem 1.3-7. *Let N and N' be subsets of Q. If N is independent and contains n elements, N' contains m elements, and N depends on N', then $n \leq m$.*

Proof. Let $S := \{s \mid$ there exists an independent set $N_{(s)} \subseteq N \cup N'$ such that $N_{(s)}$ contains n elements with s of them in $N'\}$. Then $S \neq \emptyset$, since $N \subseteq N \cup N'$, N is independent and contains n elements of which q, with $0 \leq q \leq n$, are in N'.

Let $r = maxS$, i.e. r is the largest integer such that there exists an independent set $N_{(r)} \subseteq N \cup N'$ such that $N_{(r)}$ contains n elements with r of them in N'. Then $0 \leq r \leq n$. Suppose that $r < n$. Then there exists a $w \in N_{(r)}$ such that $w \notin N'$, $w \in N$. Since $N_{(r)}$ is independent, $w \not\lhd N_{(r)} \setminus \{w\}$. Moreover, by Note 1.3-6(b), $w \lhd N'$ since N depends on N'. Hence, by (D_2), there exists a $v \in N'$ such that $v \not\lhd N_{(r)} \setminus \{w\}$. Hence, by (D_1), $v \notin N_{(r)} \setminus \{w\}$.

Next, let $\overline{N} := (N_{(r)} \setminus \{w\}) \cup \{v\}$. Then $\overline{N}$ contains n elements with $r+1$ of them in N'. Thus $\overline{N}$ is not independent. But, by Theorem 1.3-3, $N_{(r)} \setminus \{w\}$ is independent. Hence, by Theorem 1.3-4, $v \lhd N_{(r)} \setminus \{w\}$. This is a contradiction. Therefore $n = r \leq m$.

Definition 1.3-8. A subset B of Q is a *base* of Q if

(a) B is independent, and

(b) Q depends on B.

Theorem 1.3-9. *If L is an independent subset of Q, then there is a base B of Q with $L \subseteq B$.*

Proof. Let $\mathbf{E}$ be the set of all independent subsets of Q and let $\mathbf{C} := \{L\}$. Then, by Zorn's Lemma, there is a maximal chain $\mathbf{M}$ in Q with $\mathbf{C} \subseteq \mathbf{M} \subseteq \mathbf{E}$.

Let $B := \cup\{M \mid M \in \mathbf{M}\}$. Then $L \subseteq B \subseteq Q$. Let B_e be a finite subset of B. Then B_e is contained in a finite union of sets of $\mathbf{M}$. Since $\mathbf{M}$ is a chain $B_e \subseteq M_\ell$, where M_ℓ is the largest of these sets. Hence, by Theorem 1.3-3, B_e is independent. Therefore B is independent.

Suppose that Q does not depend on B. Then there exists an $x \in Q$ such that $x \not\lhd B_e$ for each finite subset B_e of B. It follows from Theorem 1.3-4 that $B_e \cup \{x\}$ is independent. Furthermore, by (D_1), $x \notin B$ - otherwise $x \lhd \{x\}$, and $\{x\}$ is a finite subset of B.

Next, let $\mathbf{M}'$ be the family of all sets of $\mathbf{M}$ together with $B \cup \{x\}$. Then $\mathbf{M}'$ is a chain in Q (for each $M \in \mathbf{M}$, $M \subseteq B \cup \{x\}$) and $\mathbf{C} \subseteq \mathbf{M} \subseteq \mathbf{M}' \subseteq \mathbf{E}$. But $\mathbf{M} \neq \mathbf{M}'$, for $B \cup \{x\} \notin \mathbf{M}$ since $x \notin B$. This contradicts the maximality of $\mathbf{M}$. Hence Q depends on B. Therefore B is a base of Q.

Theorem 1.3-10. *Let B be a base of Q with n elements. Then any other base D of Q also has n elements.*

Proof. First, we will show that D is finite. Let $B = \{x_1, x_2, \ldots, x_n\}$. Then, since Q depends on D, there exist finite subsets D_i of D such that $x_i \lhd D_i$ for $i = 1, 2, \ldots, n$.

Let $E := \cup\{D_i \mid i = 1, 2, \ldots, n\}$. Then $E \subseteq D$. To show that $E = D$, we assume that it is not so and then we deduce a contradiction from this assumption. Suppose that there is a $y \in D$, with $y \notin E$. Since Q depends on B, $y \lhd B$ (see Note 1.3-6(a)). But, by Theorem 1.3-1, $x_i \lhd E$ for $i = 1, 2, \ldots, n$. Therefore, by (D_2), $y \lhd E$. Moreover, since $y \notin E$, $(E \cup \{y\}) \setminus \{y\} = E$. Hence $y \lhd (E \cup \{y\}) \setminus \{y\}$. Therefore the finite subset $E \cup \{y\}$ of D is not independent. This contradicts the independence of D. Hence $D = E$, which contains a finite number of elements.

Suppose that D contains m elements. It must be shown that $m = n$. Let $x \in B \subseteq Q$. Then $x \lhd D$. Therefore, by Note 1.3-6(b), B depends on D. Hence, by Theorem 1.3-7, $n \leq m$. Similarly, D depends on B. Hence $m \leq n$. Therefore $m = n$.

Theorem 1.3-11. *Let B and D be bases of Q. Then B and D have the same cardinal number.*

Proof. The case of finite bases is dealt with in Theorem 1.3-10. Hence, let B and D be infinite bases with cardinal numbers n and m respectively. Let $B := \{x_\alpha \mid \alpha \in \wedge\}$. Then, since Q depends on D, there is, for each $\alpha \in \wedge$, a finite subset D_α of D such that $x_\alpha \lhd D_\alpha$. Let $E := \cup\{D_\alpha \mid \alpha \in \wedge\}$. Then $E \subseteq D$. We will now show that $E = D$. Suppose that there exists a $y \in D$, with $y \notin E$. Then, since Q depends on B, there is a finite subset B_e of B such that $y \lhd B_e$. But $B_e = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_q}\}$, with $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \subseteq \wedge$. Let $E_q := \cup\{D_{\alpha_i} \mid i = 1, 2, \ldots, q\}$. Then, by Theorem 1.3-1, $x_{\alpha_i} \lhd E_q$ for $i = 1, 2, \ldots, q$. Hence, by (D_2), $y \lhd E_q$. Moreover, since $y \notin E_q$, $(E_q \cup \{y\}) \setminus \{y\} = E_q$. Hence $y \lhd (E_q \cup \{y\}) \setminus \{y\}$. Therefore the finite subset $E_q \cup \{y\}$ of D is not independent. This contradicts the independence of D.

Since $D = \cup\{D_\alpha : \alpha \in \wedge\}$, $m \leq \aleph_0 n$, with $\aleph_0$ the cardinal number of the set of natural numbers. Furthermore, since n is infinite, $\aleph_0 n = n$. Therefore $m \leq n$. Similarly, $n \leq m$. Therefore $m = n$.

Note 1.3-12. As a consequence of Theorems 1.3-9 and 1.3-11 we define the *dimension* of Q as the cardinal number of a base of Q. The dimension of Q is denoted by $dimQ$.

Chapter 2

Theory of near vector spaces

2.1 F-groups

Definition 2.1-1. An *F-group* is a structure (V, F) which satisfies the following conditions:
(F_1) $(V, +)$ is a group and F is a set of endomorphisms of V.
(F_2) The endomorphisms $0, 1$ and -1, defined by $0x = 0$, $1x = x$ and $(-1)x = -x$ for each $x \in V$, are elements of F.
(F_3) $F^* := F \setminus \{0\}$ is a subgroup of the group of automorphisms of $(V, +)$.
(F_4) If $\alpha x = \beta x$ with $x \in V$ and $\alpha, \beta \in F$, then $\alpha = \beta$ or $x = 0$.

Note 2.1-2. (a) If $V \neq \{0\}$, then there is a $v \in V$, with $v \neq 0$. Hence $0v = 0 \neq 1v$. Consequently $0 \neq 1$.
(b) $(V, +)$ is abelian, since, by (F_2),

$$x + y = (-1)(-x) + (-1)(-y) = (-1)(-x - y) = (-1)(-(y + x)) = y + x.$$

(c) A vector space is an F-group. Other examples are given in Chapter 3.
(d) If $\alpha \in F$, then $\alpha 0 = 0$ and $\alpha(-x) = -\alpha x$ since α is an endomorphism of V.

2.2 Quasi-kernels

Definition 2.2-1. Let (V, F) be an F-group. The *quasi-kernel* $Q(V)$ (or just Q if there is no danger of confusion) of (V, F) is the set of all $u \in V$ such that, for each pair α, $\beta \in F$, there exists a $\gamma \in F$ for which

$$\alpha u + \beta u = \gamma u. \tag{1}$$

Lemma 2.2-2. *The quasi-kernel Q has the following properties:*
(a) $0 \in Q$.
(b) *For $u \in Q \setminus \{0\}$, γ in (1) is uniquely determined by α and β.*
(c) *If $u \in Q$ and $\lambda \in F$, then $\lambda u \in Q$, i.e. $Fu \subseteq Q$.*
(d) *If $u \in Q$ and $\lambda_i \in F$, $i = 1, \ldots, n$, then $\sum_{i=1}^{n} \lambda_i u = \eta u \in Q$ for some $\eta \in F$.*
(e) *If $u \in Q$ and $\alpha, \beta \in F$, then there exists a $\gamma \in F$ such that $\alpha u - \beta u = \gamma u$.*

Proof (a) $\alpha 0 + \beta 0 = 0 = \gamma 0$
(b) Let $\alpha u + \beta u = \gamma u = \gamma' u$, with $u \neq 0$. Then, by (F_4), $\gamma = \gamma'$.
(c) If $\lambda = 0$, then $0u = 0 \in Q$ when $u \in Q$. If $\lambda \neq 0$, then, by (F_3) and (1), $(\alpha\lambda)u + (\beta\lambda)u = \gamma u = \gamma\lambda^{-1}\lambda u$, so $\alpha(\lambda u) + \beta(\lambda u) = (\gamma\lambda^{-1})(\lambda u)$, which implies that $\lambda u \in Q$.
(d) We shall use induction on n. Let

$$S := \{\, n \in N \mid \sum_{i=1}^{n} \lambda_i u \in Fu \text{ if } u \in Q \text{ and } \lambda_i \in F,\ i = 1, \ldots, n\}.$$

Then, by (c), $1 \in S$. Now suppose that $k \in S$, i.e. $\eta u := \sum_{i=1}^{k} \lambda_i u \in Q$ if $u \in Q$. Then

$$\begin{aligned}\alpha(\sum_{i=1}^{k+1} \lambda_i u) &= \alpha(\sum_{i=1}^{k} \lambda_i u + \lambda_{k+1} u) \\ &= \alpha(\eta u + \lambda_{k+1} u) \\ &= \alpha(\mu u) \\ &= (\alpha\mu) u \\ &= \gamma u,\end{aligned}$$

for appropriate μ and γ in F. Hence $k+1 \in S$, and consequently $S = N$.

(e) Let $u \in Q$ and $\alpha, \beta \in F$. Then $\beta(-1) = -\beta \in F$, since $-\beta u = \beta(-u) = \beta(-1)u$ (see Note 2.1-2(d) and (F_2)). Therefore, by (1), there is a $\gamma \in F$ such that $\alpha u + (-\beta)u = \gamma u$, which implies that $\alpha u - \beta u = \gamma u$.

Definition 2.2-3. (V, F) is said to be a *linear* F-group if $V = \{0\}$ or $Q(V) \neq \{0\}$.

We shall, in what follows, associate a near field with each $u \in Q(V) \setminus \{0\}$ in a linear F-group.

Recall that a *near field* is an algebraic structure with two operations $+$ and ., which satisfy every axiom of a division ring except the right distributive law: $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$.
In the place of that distributive law, it turns out that the near field N is zero-symmetric, i.e. $0\gamma = 0$ for each $\gamma \in N$(see [5],p. 249).

Definition 2.2-4. Let (V, F) be a linear F-group, and let $u \in Q(V) \setminus \{0\}$. Define the operation $\underset{u}{+}$ on F by

$$(\alpha \underset{u}{+} \beta)u := \alpha u + \beta u \qquad (\alpha, \beta \in F). \tag{2}$$

Note 2.2-5. (a) On account of Lemma 2.2-2, $\alpha \underset{u}{+} \beta$ is uniquely determined by α and β in F.

(b) Since V is abelian, the set of all endomorphisms of V is a ring if we define addition in the following way:

$$(\alpha + \beta)x := \alpha x + \beta x.$$

In general, $\alpha + \beta$, for α and β in F, does not belong to F. It therefore differs from the sum defined in Definition 2.2-4.

Theorem 2.2-6. *Let (V, F) be a linear F-group and let $u \neq 0$ be an element of the quasi-kernel $Q(V)$. Then $(F, \underset{u}{+}, .)$ with addition $\underset{u}{+}$ as defined in Definition 2.2-4, is a near field.*

Proof. First, we shall show that $(F, \underset{u}{+})$ is an abelian group.

(i)

$$\begin{aligned}[(\alpha \underset{u}{+} \beta) \underset{u}{+} \gamma]u &= (\alpha \underset{u}{+} \beta)u + \gamma u \\ &= (\alpha u + \beta u) + \gamma u \\ &= \alpha u + (\beta u + \gamma u) \\ &= \alpha u + (\beta \underset{u}{+} \gamma)u \\ &= [\alpha \underset{u}{+} (\beta \underset{u}{+} \gamma)]u.\end{aligned}$$

Hence, by (F_4), $(\alpha \underset{u}{+} \beta) \underset{u}{+} \gamma = \alpha \underset{u}{+} (\beta \underset{u}{+} \gamma)$. Therefore $(F, \underset{u}{+})$ is associative.

(ii) By (F_2), $0 : V \to V$, defined by $0v = 0$ for each $v \in V$, is an element of F. But

$$(f \underset{u}{+} 0)u = fu + 0u = fu + 0 = fu.$$

Hence, by (F_4), $f \underset{u}{+} 0 = f$. Similarly $0 \underset{u}{+} f = f$. Therefore $0 : V \to V$ is the zero element of $(F, \underset{u}{+})$.

(iii) Define, for $f \in F$, $-f : V \to V$ by $(-f)v = f(-v)$ for each $v \in V$. Then

$$(-f \underset{u}{+} f)u = (-f)u + fu = f(-u) + fu = f(-u + u) = f0 = 0 = 0u.$$

Hence, by (F_4), $-f \underset{u}{+} f = 0$. Similarly $f \underset{u}{+} -f = 0$. Therefore, for each $f \in F$, the additive inverse, $-f$, exists and is an element of $(F, \underset{u}{+})$.

(iv) $(F, \underset{u}{+})$ is abelian:

$$\begin{aligned}(\beta \underset{u}{+} \gamma)u &= \beta u + \gamma u \\ &= \gamma u + \beta u \\ &= (\gamma \underset{u}{+} \beta)u.\end{aligned}$$

Hence, by (F_4), $\beta \underset{u}{+} \gamma = \gamma \underset{u}{+} \beta$.

Secondly, by (F_3), $(F^*, .)$ is a group.

Finally, we shall show that the left distributive law holds:

$$\begin{aligned}\alpha(\beta \underset{u}{+} \gamma)u &= \alpha(\beta u + \gamma u) \\ &= \alpha(\beta u) + \alpha(\gamma u) \\ &= (\alpha\beta)u + (\alpha\gamma)u \\ &= (\alpha\beta \underset{u}{+} \alpha\gamma)u.\end{aligned}$$

Hence, by (F_4), $\alpha(\beta \underset{u}{+} \gamma) = \alpha\beta \underset{u}{+} \alpha\gamma$.

Corollary 2.2-7. *If (V, F) is a linear F-group with $V \neq \{0\}$, then F^* is the multiplicative group of a near field.*

Theorem 2.2-8. *If $u \in Q(V) \setminus \{0\}$ and $\lambda \in F \setminus \{0\}$, then the near fields $(F, \underset{u}{+}, .)$ and $(F, \underset{\lambda u}{+}, .)$ are isomorphic.*

Proof. Define $f : (F, \underset{\lambda u}{+}) \to (F, \underset{u}{+})$ by

$$f(\alpha) = \lambda^{-1}\alpha\lambda =: \alpha^{\lambda} \quad \text{for each } \alpha \in F^*$$

and

$$f(0) = 0.$$

First, f is well defined. Let $\alpha = \beta$. Then $f(\alpha) = \lambda^{-1}\alpha\lambda = \lambda^{-1}\beta\lambda = f(\beta)$.
Next, f is bijective. Let $f(\alpha) = f(\beta)$. Then $\lambda^{-1}\alpha\lambda = \lambda^{-1}\beta\lambda$ which implies that

$$\alpha = \lambda\lambda^{-1}\alpha\lambda\lambda^{-1} = \lambda\lambda^{-1}\beta\lambda\lambda^{-1} = \beta.$$

Hence f is injective. Furthermore, let $\beta \in (F, \underset{u}{+})$. Then $\lambda\beta\lambda^{-1} = \alpha \in F$. Hence $\beta = \lambda^{-1}\alpha\lambda$. Therefore there exists an $\alpha \in (F, \underset{\lambda u}{+})$ such that $f(\alpha) = \lambda^{-1}\alpha\lambda = \beta$. Hence f is surjective.
Finally, f respects the operations.

$$\begin{aligned}
[f(\alpha \underset{\lambda u}{+} \beta)]u &= \lambda^{-1}(\alpha \underset{\lambda u}{+} \beta)(\lambda u) \\
&= \lambda^{-1}(\alpha(\lambda u) + \beta(\lambda u)) \\
&= \lambda^{-1}\alpha\lambda u + \lambda^{-1}\beta\lambda u \\
&= [\lambda^{-1}\alpha\lambda \underset{u}{+} \lambda^{-1}\beta\lambda]u \\
&= [f(\alpha) \underset{u}{+} f(\beta)]u.
\end{aligned} \tag{3}$$

Hence, by (F_4), $f(\alpha \underset{\lambda u}{+} \beta) = f(\alpha) \underset{u}{+} f(\beta)$. Therefore $(F, \underset{\lambda u}{+}) \approx (F, \underset{u}{+})$. Consequently $(F, \cdot, \underset{\lambda u}{+}) \approx (F, \cdot, \underset{u}{+})$.

Note 2.2-9. From (3) we have $f(\alpha \underset{\lambda u}{+} \beta) = f(\alpha) \underset{u}{+} f(\beta)$. Hence

$$\lambda^{-1}(\alpha \underset{\lambda u}{+} \beta)\lambda = \lambda^{-1}\alpha\lambda \underset{u}{+} \lambda^{-1}\beta\lambda.$$

This implies that

$$(\alpha \underset{\lambda u}{+} \beta)^{\lambda} = \alpha^{\lambda} \underset{u}{+} \beta^{\lambda}.$$

Therefore

$$\alpha \underset{\lambda u}{+} \beta = (\alpha^{\lambda} \underset{u}{+} \beta^{\lambda})^{\lambda^{-1}}. \tag{4}$$

Definition 2.2-10. Let (V, F) be a linear F-group with $u \in Q(V) \setminus \{0\}$. Define the *kernel* $R_u(V) = R_u$ of (V, F) by the set

$$R_u := \{v \in V \mid (\alpha \underset{u}{+} \beta)v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\}.$$

Note 2.2-11. (a) $u \in R_u$: Indeed, $u \in V$ and $(\alpha \underset{u}{+} \beta)u = \alpha u + \beta u$.

(b) $R_u \subseteq Q$: Let $v \in R_u$, then for every $\alpha, \beta \in F$ there exists a $\gamma := \alpha \underset{u}{+} \beta$ such that $\alpha v + \beta v = \gamma v$.
(c) $0 \in R_u : (\alpha \underset{u}{+} \beta)0 = 0 = \alpha 0 + \beta 0$.
(d) $(R_u, +)$ is a subgroup of $(V, +)$: Let $v, w \in R_u$. Then

$$\begin{aligned}(\alpha \underset{u}{+} \beta)(v - w) &= (\alpha \underset{u}{+} \beta)v + (\alpha \underset{u}{+} \beta)(-w) \\ &= (\alpha \underset{u}{+} \beta)v - (\alpha \underset{u}{+} \beta)w \\ &= \alpha v + \beta v - (\alpha w + \beta w) \\ &= \alpha v - \alpha w + \beta v - \beta w \\ &= \alpha(v - w) + \beta(v - w).\end{aligned}$$

Hence $v - w \in R_u$.

Theorem 2.2-12. $Q \supseteq FR_u := \{\lambda v \mid \lambda \in F, v \in R_u\}$.

Proof. Let $\lambda v \in FR_u$. Then $\lambda \in F$ and $v \in R_u$. Hence $\lambda \in F$ and $v \in Q$. Therefore, by Lemma 2.2-2(c), $\lambda v \in Q$.

Note 2.2-13. $Q = FR_u$ only in special cases (see Section 2.5).

Lemma 2.2-14. *Let u and v be elements of $Q \setminus \{0\}$. If $v \notin Fu$ and $\alpha u + \beta v = \alpha' u + \beta' v$ $(\alpha, \beta, \alpha', \beta' \in F)$, then $\alpha = \alpha'$ and $\beta = \beta'$.*

Proof. Since $u, v \in Q$, there exist, by Lemma 2.2-2(e), $\gamma, \delta \in F$ such that

$$\gamma u = \alpha u - \alpha' u \quad \text{and} \quad \delta v = \beta' v - \beta v.$$

But $\alpha u - \alpha' u = \beta' v - \beta v$. Hence $\gamma u = \delta v$. Suppose that $\delta \neq 0$. Then $v = \delta^{-1}(\gamma u) = (\delta^{-1}\gamma)u \in Fu$. This is a contradiction. Therefore $\delta = 0$. Hence $\beta v = \beta' v$. But $v \neq 0$; hence, by (F_4), $\beta = \beta'$. Similarly, $\alpha = \alpha'$.

Lemma 2.2-15. *If $v \in R_u, w, v + w \in Q$ and $w \notin Fv$, then $w \in R_u$.*

Proof. Since $v + w \in Q$, there exists, for every $\alpha, \beta \in F$, a $\gamma \in F$ such that

$$\alpha(v + w) + \beta(v + w) = \gamma(v + w).$$

Hence

$$\begin{aligned}\alpha v + \beta v + \alpha w + \beta w &= \alpha v + \alpha w + \beta v + \beta w \\ &= \gamma v + \gamma w,\end{aligned}$$

which implies that

$$(\alpha \underset{u}{+} \beta)v + \alpha w + \beta w = \gamma v + \gamma w.$$

But $w \in Q$. Therefore there exists a $\gamma' \in F$ such that $\alpha w + \beta w = \gamma' w$. Hence

$$(\alpha \underset{u}{+} \beta)v + \gamma' w = \gamma v + \gamma w,$$

which implies that

$$\alpha \underset{u}{+} \beta = \gamma = \gamma'.$$

Hence $\alpha w + \beta w = \gamma' w = \gamma w = (\alpha \underset{u}{+} \beta)w$. Therefore $w \in R_u$.

Note 2.2-16. By Note 2.2-11(d), we have that $v + w$, as in Lemma 2.2-15, is an element of R_u.

Theorem 2.2-17. *Let $Q(V)$ be the quasi-kernel of the F-group V and suppose that $u, v \in Q(V) \setminus \{0\}$ with $v \notin Fu$. Then, for any $\lambda \in F \setminus \{0\}$,*

$$u + \lambda v \in Q(V) \text{ if and only if } \underset{u}{+} = \underset{\lambda v}{+}.$$

Proof. Suppose that $u + \lambda v \in Q(V)$. By Note 2.2-11(a), $u \in R_u$, by Lemma 2.2-2(c), $v \in Q(V)$ implies that $\lambda v \in Q(V)$, and $v \notin Fu$ implies that $\lambda v \notin Fu$. Hence, by Lemma 2.2-15, we have $\lambda v \in R_u$. Therefore, for every $\alpha, \beta \in F$,

$$\begin{aligned}(\alpha \underset{u}{+} \beta)\lambda v &= \alpha\lambda v + \beta\lambda v \\ &= (\alpha \underset{\lambda v}{+} \beta)\lambda v.\end{aligned}$$

Hence $\underset{u}{+} = \underset{\lambda v}{+}$.

Conversely, suppose that $\underset{u}{+} = \underset{\lambda v}{+}$. Then $\lambda v \in R_u$. Furthermore $u \in R_u$. Hence, by Note 2.2-11(d), $u + \lambda v \in R_u$. Consequently, by Note 2.2-11(b), $u + \lambda v \in Q(V)$.

2.3 A dependence relation in $Q(V)$

Let $Q(V)$ be the quasi-kernel of the F-group V. Define a relation between $Q(V)$ and $2^{Q(V)}$ as follows:

(i) $v \triangleleft \emptyset$ if $v = 0$,

(ii) $v \triangleleft M$, $\emptyset \neq M \subseteq Q$, if there exist $u_i \in M$ and $\lambda_i \in F$ $(i = 1, \ldots, n)$ such that

$$v = \sum_{i=1}^{n} \lambda_i u_i. \tag{5}$$

Theorem 2.3-1. *Let $Q(V)$ be the quasi-kernel of the F-group V. Then the relation defined in (5) is a dependence relation between $Q(V)$ and $2^{Q(V)}$.*

Proof. Let M and N be subsets of $Q(V)$. We have to show that $(D_1), (D_2)$ and (D_3) are valid.

(D_1): Suppose that $v \in M$. Then $v = 1v$ with $1 \in F$. Hence $v \triangleleft M$.

(D_2): Suppose that $w \lhd M$ and $v \lhd N$ for each $v \in M$. Then $w = \sum_{i=1}^{n} \lambda_i u_i$, where $u_i \in M$ and $\lambda_i \in F$ $(i = 1, \ldots, n)$. But, for every i $(1 \leq i \leq n)$, $u_i = \sum_{j=1}^{m_i} \beta_{ji} w_{ji}$, where $\beta_{ji} \in F$ and $w_{ji} \in N$ $(j = 1, \ldots, m_i)$. Hence

$$w = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m_i} \beta_{ji} w_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_i \beta_{ji} w_{ji},$$

where $\lambda_i \beta_{ji} \in F$ and $w_{ji} \in N$ for every i $(1 \leq i \leq n)$ and every j $(1 \leq j \leq m_i)$. Hence $w \lhd N$.

(D_3): Suppose that $v \lhd M$ and $v \not\lhd M \setminus \{u\}$. Then $v = \sum_{i=1}^{n} \lambda_i u_i$, where $u_i \in M$ and $\lambda_i \in F$ $(i = 1, \ldots, n)$. Then, if $u \neq u_j$ for $j = 1, \ldots, n$, $v \lhd M \setminus \{u\}$. This is a contradiction. Hence $u = u_j$ for some j $(1 \leq j \leq n)$. Hence

$$v = \lambda_1 u_1 + \cdots + \lambda_{j-1} u_{j-1} + \lambda_j u_j + \lambda_{j+1} u_{j+1} + \cdots + \lambda_n u_n,$$

with $\lambda_j \neq 0$. This implies that

$$u = u_j = -\lambda_j^{-1} \lambda_1 u_1 + \cdots + -\lambda_j^{-1} \lambda_{j-1} u_{j-1} + -\lambda_j^{-1} \lambda_{j+1} u_{j+1} + \cdots + -\lambda_j^{-1} \lambda_n u_n + \lambda_j^{-1} v.$$

But, by (F_3), $\lambda_j^{-1} \in F$ for every j $(1 \leq j \leq n)$. Hence $-\lambda_j^{-1} \lambda_i \in F$, for every i $(1 \leq i \leq n)$. Therefore $u \lhd (M \setminus \{u\}) \cup \{v\}$.

Note 2.3-2. (a) By Theorem 2.3-1, the concepts and theorems of Section 1.3 are applicable to $Q(V)$.

(b) It follows directly that a set M of $Q(V)$ is independent if and only if $\sum_{i=1}^{n} \lambda_i u_i = 0$, with $u_i \in M$ and $\lambda_i \in F$ $(i = 1, \ldots, n)$, implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Thus, for $x \in Q(V)$, $x \neq 0$, $\{x\}$ is independent.

(c) A subset E of $Q(V)$ is called a generating system of $Q(V)$ if $Q(V)$ depends on E, i.e. if for every $v \in Q(V)$, there exist $\lambda_i \in F$ and $u_i \in E$ $(i = 1, \ldots, n)$ such that $v = \sum_{i=1}^{n} \lambda_i u_i$.

(d) As stated in Note 1.3-12, the dimension of $Q(V)$, denoted by $dimQ(V)$, is the cardinal number of a base of $Q(V)$.

Lemma 2.3-3. *If $\{u_i \mid i \in I\}$ is a base of $Q(V)$ and $\lambda_i \in F \setminus \{0\}$ for each $i \in I$, then $\{\lambda_i u_i \mid i \in I\}$ is a base for $Q(V)$.*

Proof. Let $v \in Q(V)$. Then

$$v = \sum_{r=1}^{n} \alpha_r u_r = \sum_{r=1}^{n} (\alpha_r \lambda_r^{-1})(\lambda_r u_r),$$

with $\alpha_r \in F$ and $u_r \in \{u_i \mid i \in I\}$ $\quad (1 \leq r \leq n)$. Hence $Q(V)$ depends on $\{\lambda_i u_i \mid i \in I\}$. Furthermore, if $\sum_{j=1}^{m} \alpha_j (\lambda_j u_j) = 0$ with $\alpha_j \in F$, $u_j \in \{u_i \mid i \in I\}$ $(1 \leq j \leq m)$, then $\sum_{j=1}^{m} (\alpha_j \lambda_j) u_j = 0$. Hence, by Note 2.3-2(b),

$$\alpha_1 \lambda_1 = \alpha_2 \lambda_2 = \cdots = \alpha_m \lambda_m = 0.$$

Therefore, since $\lambda_j \neq 0$ $(1 \leq j \leq m)$, $\alpha_j = 0$ $(1 \leq j \leq m)$. Hence, by Note 2.3-2(b), $\{\lambda_i u_i \mid i \in I\}$ is independent.

2.4 Near vector spaces

In the sequal we shall consider an F-group V which depends on its quasi-kernel $Q(V)$ (see Definition 2.4-1).

Definition 2.4-1. An F-group (V, F) is called a *near vector space* over F if the following condition holds:

$$(Q_1)\ \text{The quasi} - \text{kernel Q(V) of V generates the group (V}, +).$$

Note 2.4-2. (a) Every near vector space over F is a linear F-group.

(b) In a near vector space V with quasi-kernel $Q(V)$ a base of $Q(V)$ is called a base of V, and we define $dimV := dimQ(V)$.

(c) Every vector space is a near vector space.

(d) A near field F over itself is a near vector space, but in general not a vector space, of dimension one. This can be shown as follows. (F, F) satisfies the conditions of an F-group. Furthermore, since F is a near field it has an identity e. Moreover, $e \in Q(F)$ since $\alpha e + \beta e = (\alpha + \beta)e$ for every $\alpha, \beta \in F$. Now, let x be any element of F. Then $xe \in Q(F)$ (Lemma 2.2-2 (c)). Hence $F \subseteq Q(F)$. But $Q(F) \subseteq F$. Hence $Q(F) = F$. Therefore $Q(F)$ generates F.

The next theorem gives us an important example of a near vector space.

Theorem 2.4-3. *Let $F = (F, +, \cdot)$ be a near field and let I be a non empty index set. Then the set*

$$F^{(I)} := \{(\xi_i)_{i \in I} \mid \xi_i \in F, \xi_i \neq 0 \textit{ for only a finite number of } i \in I\}$$

is a near vector space, if we define addition and multiplication component wise as follows:

$$(\xi_i) + (\eta_i) := (\xi_i + \eta_i)$$

and

$$\lambda(\xi_i) := (\lambda \xi_i), \tag{6}$$

where ξ_i, η_i and λ are elements of F and (ξ_i) is used as an abbreviation for $(\xi_i)_{i \in I}$.

Proof. First, we shall show that $(F^{(I)}, F)$ is an F-group.

(F_1): $(F^{(I)}, +)$ is a group. Since

(i)

$$\begin{aligned}
[(\xi_i) + (\eta_i)] + (\alpha_i) &= (\xi_i + \eta_i) + (\alpha_i) \\
&= ([\xi_i + \eta_i] + \alpha_i) \\
&= (\xi_i + [\eta_i + \alpha_i]) \\
&= (\xi_i) + (\eta_i + \alpha_i) \\
&= (\xi_i) + [(\eta_i) + (\alpha_i)],
\end{aligned}$$

(ii) the identity, (0), is an element of $(F^{(I)}, +)$, and
(iii) for each $(\xi_i) \in F^{(I)}$ there exists a $(-\xi_i) \in F^{(I)}$ such that $(\xi_i)+(-\xi_i) = (\xi_i - \xi_i) = (0)$.
Furthermore F is a set of endomorphisms of $F^{(I)}$ since, for each $\lambda \in F$,

$$\begin{aligned}\lambda[(\xi_i) + (\eta_i)] &= \lambda(\xi_i + \eta_i) \\ &= (\lambda[\xi_i + \eta_i]) \\ &= (\lambda\xi_i + \lambda\eta_i) \\ &= (\lambda\xi_i) + (\lambda\eta_i) \\ &= \lambda(\xi_i) + \lambda(\eta_i).\end{aligned}$$

(F_2): The endomorphisms $0, 1$ and -1, defined by

$$0(\xi_i) = (0\xi_i) = (0),$$

$$1(\xi_i) = (1\xi_i) = (\xi_i)$$

and

$$(-1)(\xi_i) = ((-1)\xi_i) = (-\xi_i),$$

with $(\xi_i) \in F^{(I)}$, are elements of F.
(F_3): F^* is a subgroup of the automorphism group of $(F^{(I)}, +)$. This can be shown in the following way. Let $\lambda \in F^*$ and (ξ_i), $(\eta_i) \in F^{(I)}$. Then λ is a bijection. Let $\lambda(\xi_i) = \lambda(\eta_i)$. Then

$$(\lambda\xi_i) = (\lambda\eta_i).$$

Hence, for each $i \in I$,

$$\lambda\xi_i = \lambda\eta_i,$$

and so

$$\lambda[\xi_i - \eta_i] = 0.$$

Therefore, for each $i \in I$, $\xi_i = \eta_i$, which implies that $(\xi_i) = (\eta_i)$. Hence λ is an injection. Now, let (ξ_i) be an element of $F^{(I)}$. Then, since $\lambda \neq 0$, $\lambda^{-1} \in F$. Hence $\lambda^{-1}\xi_i \in F$ for each $i \in I$. Therefore $(\lambda^{-1}\xi_i) \in F^{(I)}$ and $\lambda(\lambda^{-1}\xi_i) = (\lambda\lambda^{-1}\xi_i) = (\xi_i)$. Hence λ is a surjection. Consequently F^* is a subset of the automorphism group of $(F^{(I)}, +)$. But F is a near field. Hence F^* is a subgroup of the automorphism group of $(F^{(I)}, +)$.
(F_4): Let $\lambda(\xi_i) = \mu(\xi_i)$. Then $(\lambda\xi_i) = (\mu\xi_i)$, which implies that $\lambda\xi_i = \mu\xi_i$ for each $i \in I$. Suppose that $\lambda \neq \mu$. Then, if there exists a $j \in I$ such that $\xi_j \neq 0$, $\xi_j^{-1} \in F$. Hence $\lambda\xi_j\xi_j^{-1} = \mu\xi_j\xi_j^{-1}$. Therefore $\lambda = \mu$. This is a contradiction. Hence $\xi_i = 0$ for each $i \in I$.

Finally we must show that $Q(F^{(I)})$ generates $F^{(I)}$. Let $e_j := (\delta_{ji})_{i\in I} = (\delta_{ji})$,where δ_{ji} is the Kronecker symbol. Then, for every $\alpha, \beta \in F$,

$$\begin{aligned}\alpha e_j + \beta e_j &= (\alpha\delta_{ji}) + (\beta\delta_{ji})\\ &= (\alpha\delta_{ji} + \beta\delta_{ji})\\ &= ((\alpha+\beta)\delta_{ji})\\ &= (\alpha+\beta)(\delta_{ji})\\ &= (\alpha+\beta)e_j.\end{aligned}$$

Hence $\{e_j \mid j \in I\} \subseteq Q(F^{(I)})$.

Definition 2.4-4. Let F be a near field. Define the *kernel* $K(F) = K$ by the set

$$K := \{\kappa \in F \mid (\xi+\eta)\kappa = \xi\kappa + \eta\kappa \text{ for every } \xi, \eta \in F\}.$$

Theorem 2.4-5. *Let K be the kernel of a near field F. Then*

(a) K, with the operations of F, is a division ring, and

(b) F is a right vector space over K.

Proof. (a) Since $1 \in F$ and $(\xi+\eta)1 = \xi+\eta = \xi 1 + \eta 1$ for every $\xi, \eta \in F$, $\emptyset \neq K \subseteq F$. Let κ and κ_1 be elements of K. Then

$$\begin{aligned}(\xi+\eta)(\kappa-\kappa_1) &= (\xi+\eta)\kappa - (\xi+\eta)\kappa_1\\ &= \xi\kappa + \eta\kappa - (\xi\kappa_1 + \eta\kappa_1)\\ &= \xi\kappa - \xi\kappa_1 + \eta\kappa - \eta\kappa_1\\ &= \xi(\kappa-\kappa_1) + \eta(\kappa-\kappa_1).\end{aligned}$$

Hence $\kappa - \kappa_1 \in K$. Therefore K is a subgroup of $(F, +)$. Hence $(K, +)$ is an abelian group. Furthermore, $(K^*, \cdot)$ is a subgroup of $(F^*, \cdot)$. Let $\kappa \in K$ with $\kappa \neq 0$, and consider κ^{-1}. Now

$$\begin{aligned}[(\xi+\eta)\kappa^{-1}]\kappa &= (\xi+\eta)\\ &= (\xi\kappa^{-1})\kappa + (\eta\kappa^{-1})\kappa\\ &= (\xi\kappa^{-1} + \eta\kappa^{-1})\kappa,\end{aligned}$$

which implies that

$$([(\xi+\eta)\kappa^{-1}] - [\xi\kappa^{-1} + \eta\kappa^{-1}])\kappa = 0.$$

Hence, since $\kappa \neq 0$,

$$(\xi+\eta)\kappa^{-1} = \xi\kappa^{-1} + \eta\kappa^{-1}.$$

Hence $\kappa^{-1} \in K$. Moreover,

$$\begin{aligned}(\xi+\eta)\kappa\kappa_1 &= (\xi\kappa + \eta\kappa)\kappa_1\\ &= (\xi\kappa)\kappa_1 + (\eta\kappa)\kappa_1\\ &= \xi(\kappa\kappa_1) + \eta(\kappa\kappa_1).\end{aligned}$$

Hence $\kappa\kappa_1 \in K$. Therefore $(K^*, \cdot)$ is a subgroup of $(F^*, \cdot)$.
Finally,

$$\xi(\eta + \kappa) = \xi\eta + \xi\kappa$$

and

$$(\xi + \eta)\kappa = \xi\kappa + \eta\kappa,$$

for every $\xi, \eta, \kappa \in K$. Hence K is a division ring.

(b) Let $\xi, \eta \in F$ and $\kappa, \kappa' \in K$. Then
(i) $(F, +)$ is a abelian group,
(ii) $\kappa\xi \in F$,
(iii) $(\xi + \eta)\kappa = \xi\kappa + \eta\kappa$,
(iv) $\xi(\kappa + \kappa') = \xi\kappa + \xi\kappa'$,
(v) $\xi(\kappa\kappa') = (\xi\kappa)\kappa'$, and
(vi) $\xi 1 = \xi$.

Corollary 2.4-6. *Let K be the kernel of a near field F . Then $F = K$ if and only if F is a division ring.*

Proof. Let $F = K$, then F is a division ring since K is a division ring.
Conversely, let F be a division ring. Suppose that $\xi \in F$. Then $(\eta + \eta_1)\xi = \eta\xi + \eta_1\xi$ for every $\eta, \eta_1 \in F$. Hence $\xi \in K$.
We are now in a position to give an exact description of the quasi-kernel $Q(F^{(I)})$.

Theorem 2.4-7. *Consider the F-group $(F^{(I)}, F)$. Then*

$$Q(F^{(I)}) = \{\lambda(\kappa_i)_{i\in I} \mid \lambda \in F \quad \text{and} \quad \kappa_i \in K(F)\}.$$

Proof. Let $\alpha, \beta \in F$ and $\kappa_i \in K(F)$. Then

$$\begin{aligned}
\alpha(\kappa_i) + \beta(\kappa_i) &= (\alpha\kappa_i) + (\beta\kappa_i) \\
&= (\alpha\kappa_i + \beta\kappa_i) \\
&= ([\alpha + \beta]\kappa_i) \qquad (\kappa_i \in K(F)) \\
&= [\alpha + \beta](\kappa_i).
\end{aligned}$$

Hence $(\kappa_i) \in Q(F^{(I)})$. Therefore, by Lemma 2.2-2(c), $\lambda(\kappa_i) \in Q(F^{(I)})$.
Conversely, let $(\xi_i) \in Q(F^{(I)})$. If $(\xi_i) = (0)$, then $(\xi_i) = (0) = \lambda(\kappa_i)$, with $\lambda = 0$. Hence, suppose that $(\xi_i) \neq (0)$, i.e. there exists an $i_0 \in I$ such that $\xi_{i_0} \neq 0$. Let $\kappa_i := \xi_{i_0}^{-1}\xi_i$ for each $i \in I$. Then we have $(\xi_i) = \xi_{i_0}(\kappa_i)$. Hence, by Lemma 2.2-2(c), $(\kappa_i) \in Q(F^{(I)})$. Therefore there exists a $\gamma \in F$ such that, for each $i \in I$,

$$\alpha\kappa_i + \beta\kappa_i = \gamma\kappa_i. \qquad (7)$$

But $\kappa_{i_0} = \xi_{i_0}^{-1}\xi_{i_0} = 1$. Hence, since (7) holds for each $i \in I$, $\alpha 1 + \beta 1 = \gamma 1$, which implies that $\alpha + \beta = \gamma$. Therefore $\kappa_i \in F$ and $\alpha\kappa_i + \beta\kappa_i = (\alpha + \beta)\kappa_i$ for each $i \in I$. Consequently $\kappa_i \in K(F)$ for each $i \in I$.

In the next theorem we shall show how the space $F^{(I)}$ can be characterized as an F-group. First, we need to prove the following lemma.

Lemma 2.4-8. *Let V be a near vector space and let $B = \{u_i \mid i \in I\}$ be a base of $Q(V)$. Then each $x \in V$ is a unique linear combination of elements of B, i.e. there exist $\xi_i \in F$, with $\xi_i \neq 0$ for only a finite number of $i \in I$, which are uniquely determined by x and B such that*

$$x = \sum_{i \in I} \xi_i u_i.$$

Proof. Let $x \in V$. By (Q_1) there exist $v_1, \ldots, v_n \in Q(V)$ such that $x = \sum_{j=1}^{n} v_j$. Since each v_j is a linear combination of elements of B, x is also a linear combination of elements of B.
To prove the uniqueness, let

$$\sum_{i \in I} \xi_i u_i = \sum_{i \in I} \xi_i' u_i.$$

with only a finite number of the ξ_i' and ξ_i not zero. Since $B \subseteq Q$, $u_i \in Q$ $(i \in I)$. Hence, by Lemma 2.2-2(e), there are $\eta_i \in F$ $(i \in I)$ such that $\xi_i u_i - \xi_i' u_i = \eta_i u_i$ for all $i \in I$. But $\sum_{i \in I}(\xi_i u_i - \xi_i' u_i) = 0$, which implies that $\sum_{i \in I} \eta_i u_i = 0$. Therefore, since B is independent, we have $\eta_i = 0$ for all $i \in I$. Hence, for each $i \in I$,

$$\xi_i u_i - \xi_i' u_i = 0$$

and so

$$\xi_i u_i = \xi_i' u_i.$$

Therefore, for each $i \in I$, $\xi_i = \xi_i'$ since $u_i \neq 0$ for each $i \in I$.

Theorem 2.4-9. *Let V be an F-near vector space. Then there exist an index set I and a bijection $f : V \to F^{(I)}$ which respects the multiplication operation, i.e.*

$$f(\alpha x) = \alpha f(x) \quad (\alpha \in F, x \in V),$$

where $\alpha f(x)$ is defined as in (6).

Proof. Take any base B of V as index set and define f by

$$f(x) = f(\sum_{u \in B} \xi_u u) := (\xi_u)_{u \in B}.$$

Then f is well defined. This can be shown in the following way. Let x and y be elements of V. Then there are ξ_u $(u \in B)$ and η_u $(u \in B)$ such that

$$x = \sum_{u \in B} \xi_u u \quad \text{and} \quad y = \sum_{u \in B} \eta_u u.$$

Suppose that $x = y$, i.e. $\sum_{u\in B}\xi_u u = \sum_{u\in B}\eta_u u$. Then, by Lemma 2.4-8, we have $\xi_u = \eta_u$ for all $u \in B$. Hence $(\xi_u)_{u\in B} = (\eta_u)_{u\in B}$. Therefore $f(x) = f(y)$.

We shall now show that f is a bijection. Let $f(x) = f(y)$. Then

$$f(\sum_{u\in B}\xi_u u) = f(\sum_{u\in B}\eta_u u),$$

which implies that $(\xi_u)_{u\in B} = (\eta_u)_{u\in B}$. Hence $\xi_u = \eta_u$ for all $u \in B$. Therefore

$$x = \sum_{u\in B}\xi_u u = \sum_{u\in B}\eta_u u = y.$$

Hence f is injective. Furthermore, to show that f is surjective, let $(\xi_u)_{u\in B}$ be an element of $F^{(B)}$. Then $\xi_u \in F$ for all $u \in B$. Let $x = \sum_{u\in B}\xi_u u$. Then, since V is an F-near vector space, $x \in V$ and

$$f(x) = f(\sum_{u\in B}\xi_u u) = (\xi_u)_{u\in B}.$$

Finally, it can be shown that f respects the multiplication operation. Let $x \in V$, then there are ξ_u $(u \in B)$ such that $x = \sum_{u\in B}\xi_u u$. Hence

$$\begin{aligned} f(\alpha x) &= f(\alpha\sum_{u\in B}\xi_u u) \\ &= f(\sum_{u\in B}(\alpha\xi_u)u) \\ &= (\alpha\xi_u)_{u\in B} \\ &= \alpha(\xi_u)_{u\in B} \\ &= \alpha f(\sum_{u\in B}\xi_u u) \\ &= \alpha f(x). \end{aligned}$$

Note 2.4-10. Let $\{u_i \mid i \in I\}$ be a base of $Q(V)$ and let f be as defined in Theorem 2.4-9. Then $f(u_i) = (\delta_{ij})_{j\in I} =: e_i$ where δ_{ij} is the Kronecker symbol.

For further investigation of the structure of a near vector space, we introduce the following relation.

Definition 2.4-11. The elements u and v of $Q(V)\setminus\{0\}$ are called *compatible* ($u\ cp\ v$), if there is a $\lambda \in F\setminus\{0\}$ such that $u + \lambda v \in Q$.

Lemma 2.4-12. *The elements u and v of $Q(V)\setminus\{0\}$ are compatible if and only if there exists a $\lambda \in F\setminus\{0\}$ such that* $\underset{u}{+} = \underset{\lambda v}{+}$.

Proof. If $v \notin Fu$, the theorem follows immediately from Theorem 2.2-17. Suppose that $v \in Fu$, i.e. $v = \alpha u$ for a $\alpha \in F\setminus\{0\}$. Then both statements $u\ cp\ v$ and $\underset{u}{+} = \underset{\lambda v}{+}$ hold true for $u, v \in Q(V)\setminus\{0\}$ and $\lambda \in F\setminus\{0\}$. This can be shown as follows:

(i)

$$u \; cp \; \alpha u \tag{8}$$

since, by Lemma 2.2-2(d), $1u + \lambda\alpha u \in Q(V)$ for each $\lambda \in F$.
(ii) $\underset{u}{+} = \underset{\lambda v}{+}$, since $\lambda v = \alpha^{-1}\alpha u = u$ if we take $\lambda = \alpha^{-1}$.

Theorem 2.4-13. *The compatibility relation cp is an equivalence relation on* $Q(V) \setminus \{0\}$.

Proof. (i) The relation cp is reflexive since, by Lemma 2.2-2(d), $u + \lambda u \in Q$ and therefore $u \; cp \; u$.
(ii) The relation cp is symmetric. This can be shown as follows. Let $u \; cp \; v$. Then

$$u + \lambda v \in Q.$$

Hence, by Lemma 2.2-2(c),

$$\lambda^{-1}(u + \lambda v) \in Q.$$

Therefore

$$\lambda^{-1}u + v = v + \lambda^{-1}u \in Q.$$

Hence

$$v \; cp \; u.$$

(iii) The relation cp is transitive. This can be shown as follows. Let $u \; cp \; v$ and $v \; cp \; w$. Then, by Lemma 2.4-12, $\underset{u}{+} = \underset{\lambda v}{+}$ and $\underset{v}{+} = \underset{\mu w}{+}$. It suffices to show that $\underset{u}{+} = \underset{\eta w}{+}$. Now

$$\begin{aligned} \alpha \underset{u}{+} \beta &= \alpha \underset{\lambda v}{+} \beta \\ &= (\alpha^{\lambda} \underset{v}{+} \beta^{\lambda})^{\lambda^{-1}} \qquad (\text{by}(4)) \\ &= (\alpha^{\lambda} \underset{\mu w}{+} \beta^{\lambda})^{\lambda^{-1}} \\ &= \alpha \underset{\lambda\mu w}{+} \beta \\ &= \alpha \underset{\eta w}{+} \beta. \end{aligned}$$

Hence, by Lemma 2.4-12, $u \; cp \; w$.

Theorem 2.4-14. *Let u, v and $u + v$ be elements of $Q \setminus \{0\}$. Then*
(a) $u \; cp \; v$, and
(b) $u \; cp \; u + v$.

Proof. (a) Since u and v are elements of Q, $u \; cp \; v$ follows from Definition 2.4-11 by putting $\lambda = 1$.

(b) (i) If $v \in Fu$, i.e. $v = \alpha u$ with $\alpha \in F \setminus \{0\}$, then $u \; cp \; u + v$ since $u + 1(u + v) = u + 1(u + \alpha u) \in Q$ by Lemma 2.2-2(d).
(ii) If $v \notin Fu$, then, by Lemma 2.2-15, $u + v \in R_u$. Hence, by Note 2.2-11, $u + (u + v) \in R_u \subseteq Q$. Therefore $u \; cp \; u + v$.

Definition 2.4-15. A near vector space V is called a *regular near vector space* if the following condition holds:

(Q_2) Any two vectors of $Q(V) \setminus \{0\}$ are compatible.

Theorem 2.4-16. *A near vector space V is regular if and only if there exists a base which consists of mutually compatible vectors.*

Proof. Suppose that V is regular. Then, by (Q_2), any two vectors of $Q(V) \setminus \{0\}$ are compatible. Therefore every base of $Q(V)$ (by Note 2.4-2(b) also of V) consists of mutually compatible vectors.

Conversely, let V be a near vector space with a base B consisting of mutually compatible vectors. Let $u \in Q(V) \setminus \{0\}$. Then, by Lemma 2.4-8, u can be written as a linear combination of base elements. Therefore, without loss of generality, $u = \sum_{i=1}^{r} \lambda_i u_i$ with $u_i \in B$ and $\lambda_i \neq 0$ for all $i \in \{1, \dots, r\}$. Let

$$u' := \begin{cases} \sum_{i=1}^{r-1} \lambda_i u_i & \text{for } r > 1 \\ 0 & \text{for } r = 1. \end{cases}$$

Then $u = u' + \lambda_r u_r \in Q(V)$. Hence, for every $\alpha, \beta \in F$, there exist a $\gamma \in F$ such that

$$\alpha(u' + \lambda_r u_r) + \beta(u' + \lambda_r u_r) = \alpha u + \beta u = \gamma u = \gamma(u' + \lambda_r u_r).$$

Hence

$$\alpha u' + \alpha\lambda_r u_r + \beta u' + \beta\lambda_r u_r = \gamma u' + \gamma\lambda_r u_r,$$

and therefore

$$\alpha u' + \beta u' + \alpha\lambda_r u_r + \beta\lambda_r u_r = \gamma u' + \gamma\lambda_r u_r.$$

But $u_r \notin \{u_1, \dots, u_{r-1}\}$. Hence, by the uniqueness part of Lemma 2.4-8, we have that

$$\alpha\lambda_r u_r + \beta\lambda_r u_r = \gamma\lambda_r u_r.$$

Therefore

$$\alpha u' + \beta u' = \gamma u',$$

which implies that

$$u' \in Q(V).$$

Now, we shall show that u and u_r are compatible. If

(i) $u' = 0$ then $u = \lambda_r u_r$ and hence, by (8), we have $u_r \; cp \; \lambda_r u_r$;

(ii) $u' \neq 0$ then, by Theorem 2.4-14, $\lambda_r u_r \, cp \, u$ since u', $\lambda_r u_r$ and $u = u' + \lambda_r u_r = \lambda_r u_r + u'$ are elements of $Q(V)$. But, by (8), $u_r \; cp \; \lambda_r u_r$. Consequently, by Theorem 2.4-13, we have $u_r \; cp \; u$.

But, by the assumption, u_r is compatible with every other vector of B. Therefore it follows from the transitivity of cp (Theorem 2.4-13), that u is compatible with every other vector of B. Hence, since $u \in Q(V) \setminus \{0\}$ is chosen arbitrary, the following statement is valid:

If $v, w \in Q(V) \setminus \{0\}$, then $v\ cp\ u_i$ and $w\ cp\ u_i$, with $u_i \in B$.

Hence, by Theorem 2.4-13, $v\ cp\ w$. Therefore each two elements of $Q(V) \setminus \{0\}$ are compatible. Consequently V is regular.

Theorem 2.4-17 (The decomposition theorem). *Every near vector space V is the direct sum of regular near vector spaces V_j $(j \in J)$ such that each $u \in Q(V) \setminus \{0\}$ lies in precisely one direct summand V_j. The subspaces V_j are maximal regular near vector spaces.*

Proof. First, we shall show that V is the direct sum of regular near vector spaces $V_j (j \in J)$.

Let $Q \setminus \{0\}$ be decomposed in sets $Q_j (j \in J)$ of mutually compatible vectors. This is possible by Theorem 2.4-13. Furthermore, let $B \subseteq Q \setminus \{0\}$ be a base of V and let $B_j := B \cap Q_j$. The B_j's are mutually disjoint and each is an independent subset of B.

Furthermore $B = \cup_{j \in J} B_j$. This can be shown as follows. We know that $Q \setminus \{0\} = \cup_{j \in J} Q_j$. Hence

$$\cup_{j \in J} B_j = \cup_{j \in j} (B \cap Q_j) = B \cap \cup_{j \in J} Q_j = B \cap (Q \setminus \{0\}) = B.$$

Let $B = \{b_i \mid i \in I\}$, with I an index set. Since $B = \cup_{j \in J} B_j$, with the B_j's mutually disjoint, we have that, for each $i \in I$, $b_i \in B_j$ for some $j \in J$. Hence , for all $j \in J$, $B_j = \{b_{ij} := b_i \mid i \in I_j\}$ with $I = \cup_{j \in J} I_j$.

Let $V_j :=< B_j >$ be the subspace of V generated by B_j. By Theorem 2.4-16, V_j is regular since $B_j (\subseteq Q_j)$, which consists of mutually compatible vectors, is a base of V_j.

Let $x \in V$. Then, by Lemma 2.4-8, $x = \sum_{i \in I} \xi_i b_i$, with $b_i \in B$ and $\xi_i \neq 0$ for only a finite number of $i \in I$. But for each $i \in I$, $b_i \in B_j$, for some $j \in J$. Hence $x = \sum_{j \in J} (\sum_{i \in I_j} \xi_{ij} b_{ij})$ with $b_{ij} \in B_j$. Moreover, since $V_j =< B_j >$, there is an $x_j \in V_j$ such that $x_j = \sum_{i \in I_j} \xi_{ij} b_{ij}$. Hence

$$x = \sum_{j \in J} x_j. \tag{9}$$

By Theorem 2.4-8, $x = \sum_{i \in I} \xi_i b_i$ can be written in a unique way. Apply Theorem 2.4-8 to the near vector space V_j, with base B_j, for all $j \in J$. Then, for each $j \in J$, there exists an $x_j \in V_j$ which correspond uniquely to $\sum_{i \in I_j} \xi_{ij} b_{ij}$. Hence $x = \sum_{j \in J} x_j$ is uniquely determined. The existence of the direct sum $V = \sum_{j \in J} V_j$ is therefore established.

Secondly, we will show that each $u \in Q(V) \setminus \{0\}$ lies in precisely one direct summand V_j. Suppose that there exist elements in $Q(V) \setminus \{0\}$ which are not elements of V_j $(j \in J)$. Let u be such an element with the least possible number of summands in the decomposition given by (9), i.e.

$$u = \sum_{j \in J} u_j, \tag{10}$$

with $u_j \in V_j$ $(j \in J)$ and with the number of $u_j \neq 0$ $(j \in J)$ as small as possible.

Since $u \in Q$ it follows that, for every $\alpha, \beta \in F$, there exists a $\gamma \in F$ such that

$$\alpha u + \beta u = \gamma u.$$

But

$$\begin{aligned}\alpha u + \beta u &= \alpha \sum_{j \in J} u_j + \beta \sum_{j \in J} u_j \\ &= \sum_{j \in J} \alpha u_j + \sum_{j \in J} \beta u_j \\ &= \sum_{j \in J} (\alpha u_j + \beta u_j),\end{aligned}$$

and

$$\gamma u = \gamma \sum_{j \in J} u_j = \sum_{j \in J} \gamma u_j.$$

Hence

$$\sum_{j \in J} (\alpha u_j + \beta u_j) = \sum_{j \in J} \gamma u_j.$$

But, since $\sum_{j \in J} V_j$ is a direct sum, $V_i \cap V_j = \{0\}$ if $i \neq j$. Hence $\alpha u_j + \beta u_j = \gamma u_j$ for all $j \in J$. This implies that

$$\sum_{j \in J'} (\alpha u_j + \beta u_j) = \sum_{j \in J'} \gamma u_j \quad \text{for all} \quad J' \subseteq J.$$

Hence

$$\alpha \sum_{j \in J'} u_j + \beta \sum_{j \in J'} u_j = \gamma \sum_{j \in J'} u_j \quad \text{for all} \quad J' \subseteq J. \tag{11}$$

Consequently,

$$u' := \sum_{j \in J'} u_j \in Q. \tag{12}$$

Let J_u be the set of all the $j \in J$ for which $u_j \neq 0$ in the decomposition (10). Since J_u is finite and $u \notin V_j$ for some $j \in J$, $| J_u | > 1$. Furthermore, by the definition of u, $| J_{u^*} | \geq | J_u |$ for all $u^* \in Q \setminus \cup_{j \in J} V_j$.

It can be shown that if $J' \subseteq J$, is such that $J_u \cap (J \setminus J') \neq \emptyset$, then $| J_{u'} | = 1$ with u' as defined in (12). Suppose that $| J_{u'} | > 1$ ($| J_{u'} | \neq 0$, since $u' \neq 0$), then $u' = u_{j_1} + u_{j_2} + \cdots + u_{j_n}$, $n > 1$ and $J' = \{j_1, \ldots, j_n\}$. Then $u' \notin V_{j_i}$, with $j_i \in J'$, since $u' \in V_{j_i}$ implies that $u' - u_{j_i} \in V_{j_i} \cap \sum_{j \in J \setminus \{j_i\}} V_j = \{0\}$. But then $u' = u_{j_i}$ which is contradictory to our assumption. Moreover, $u' \notin V_{j'}$, with $j' \in J \setminus J'$, since $u' \in V_{j'}$ implies that $u' \in V_{j'} \cap \sum_{j \in J \setminus \{j'\}} V_j = \{0\}$, which is a contradiction. Hence $u' \notin \cup_{j \in J} V_j$. Therefore $u' \in Q \setminus \cup_{j \in J} V_j$. Hence $| J_{u'} | \geq | J_u |$. This is contradictory to our assumption that $J' (\subseteq J)$ is such that $J_u \cap (J \setminus J') \neq \emptyset$. Hence $J_{u'} = \{j'\}$ for some $j' \in J$.

Also $| J_{u-u'} | = 1$, which can be shown as follows. Suppose that $| J_{u-u'} | = m$ with $m > 1$ ($| J_{u-u'} | \neq 0$ since $J_u \cap (J \setminus J') \neq \emptyset$). Then $u - u' = u_{j_1} + \cdots + u_{j_m}$. As shown in the above paragraph, $u - u' \notin \cup_{j\in J} V_j$. Furthermore, by (11), $u - u' \in Q$. Hence $u - u' \in Q \setminus \cup_{j \in J} V_j$. Therefore $| J_{u-u'} | \geq | J_u |$. This contradicts $J_{u'} \neq \emptyset$. Hence $J_{u-u'} = \{j''\}$ for some $j'' \in J$, with $j'' \neq j'$. [Suppose that $j := j' = j''$. Then $u' = u_j$ and $u - u' = u_j$. Hence $u = 2u_j \in V_j$, which is a contradiction.]

We therefore obtain the following:

$$u = u' + (u - u'),$$

with $u' \in V_{j'}$ and $u - u' \in V_{j''}$. But $u' \in Q$ and $u - u' \in Q$. Hence $u' \in Q \cap V_{j'} =: Q_{j'}$ and $u - u' \in Q \cap V_{j''} =: Q_{j''}$. But, by Theorem 2.4-14, u' *cp* $u - u'$ since u', $u - u'$ and $u' + (u - u') \in Q$. Hence $j' = j''$. This is a contradiction. Therefore

$$Q \subseteq \cup_{j \in J} V_j.$$

Hence each $u \in Q \setminus \{0\}$ is contained in at least one V_j. Since $V_j \cap V_{j'} = \{0\}$, if $j \neq j'$, each u is contained in precisely one V_j.

Finally, we will show, by means of a contradiction, that the subspaces V_j are maximal regular near vector spaces. Suppose that there exist a $j_0 \in J$ and a regular subspace W such that $W \supset V_{j_0}$. Suppose that $Q(V_{j_0}) = Q(W)$. Then, since V_{j_0} is generated by $Q(V_{j_0})$ and W is generated by $Q(W)$, we have that $V_{j_0} = W$, which is impossible. Hence, there exists a $u \in Q \cap (W \setminus V_{j_0})$. Since $u \in Q \setminus \{0\}$, $u \in V_j$ for some $j \in J \setminus \{j_0\}$. But W is regular. Hence, since $V_{j_0} \subset W$, u is compatible with each $v \in Q(V_{j_0}) \setminus \{0\}$. This contradicts the fact that $j \neq j_0$.

Theorem 2.4-18 (The uniqueness theorem). *There exists only one direct decomposition of a near vector space into maximal regular subspaces.*

Proof. The existence of such a decomposition was shown in the previous theorem. The uniqueness will be shown as follows. Let

$$V = \sum_{j \in J} V_j = \sum_{j' \in J'} V'_{j'}, \tag{13}$$

be two direct decompositions of V in maximal regular subspaces V_j $(j \in J)$ and $V'_{j'}$ $(j' \in J')$, respectively.

Furthermore, let $Q_j := (Q(V) \setminus \{0\}) \cap V_j$. By (Q_1), $V_j = < Q_j >$. Now, each two vectors in Q_j are, by (Q_2), compatible. But Q_j is not properly contained in a set of mutually compatible vectors. This can be shown as follows. Suppose that there exists a $u \in Q(V) \setminus Q_j$ such that u *cp* v for all $v \in Q_j$. Let $Q(W_j) \setminus \{0\}$ be an equivalence class (with respect to *cp*), with $u \in Q(W_j) \setminus \{0\}$. Then $Q_j \subset Q(W_j) \setminus \{0\}$. Let $W_j := < Q(W_j) \setminus \{0\} >$. Then W_j is regular since any two elements of $Q(W_j) \setminus \{0\}$ are compatible. But $W_j \supset V_j$. This contradicts the maximality of V_j.

Moreover, every $V'_{j'}$ $(j' \in J')$ is maximal regular and so $Q'_{j'}$ is not properly contained in a set of mutually compatible vectors and therefore corresponds to a Q_j $(j \in J)$. Hence $Q_j \subseteq V'_{j'}$ and therefore $V_j \subseteq V'_{j'}$. But V_j is maximal regular and so $V_j = V'_{j'}$. Therefore

$\{V'_{j'} \mid j' \in J'\} \subseteq \{V_j \mid j \in J\}$. By symmetry, $\{V_j \mid j \in J\} \subseteq \{V'_{j'} \mid j' \in J'\}$. Consequently $\{V'_{j'} \mid j' \in J'\} = \{V_j \mid j \in J\}$. Therefore both sides of (13) contain the same subspaces.

Definition 2.4-19. The uniquely determined direct decomposition of a near vector space V in maximal regular subspaces, is called the *canonical* direct decomposition of V.

The next theorem gives some additional characteristics of the canonical direct decomposition of V.

Theorem 2.4-20. *A direct decomposition*

$$V = \sum_{j \in J} V_j \tag{14}$$

of a near vector space V in regular subspaces $V_j (j \in J)$ is canonical if and only if

$$Q(V) \subseteq \cup_{j \in J} V_j. \tag{15}$$

Proof. Suppose that a direct decomposition $V = \sum_{j \in J} V_j$ of a near vector space V in regular subspaces $V_j (j \in J)$ is canonical. Then, in the proof of Theorem 2.4-17, we have shown that $Q(V) \subseteq \cup_{j' \in J'} V'_{j'}$, where $V = \sum_{j' \in J'} V'_{j'}$ is a direct decomposition of V in maximal regular subspaces $V'_{j'}$. Hence, by Theorem 2.4-18, $\{V'_{j'} \mid j' \in J'\} = \{V_j \mid j \in J\}$. Therefore $Q(V) \subseteq \cup_{j \in J} V_j$.

Conversely, let $Q(V) \subseteq \cup_{j \in J} V_j$ and suppose that there exists a V_{j_0} in (14) which is not maximal regular, i.e. $V_{j_0} \subset W$, where, by Zorn's lemma, we can assume, without loss of generality, that W is maximal regular in V. Then there exists an $x \in Q(V) \cap (W \setminus V_{j_0})$ (see proof of Theorem 2.4-17). By (15), there exists a $j_1 \in J$ such that $x \in V_{j_1}$, with $j_1 \neq j_0$. Also $V_{j_1} + W$ is regular. This can be shown as follows. Let $u \in V_{j_1} \cap W$ and $u \in Q(V) \setminus \{0\}$. Then, for any $v \in V_{j_1} \cap (Q(V) \setminus \{0\})$, u and v are compatible since V_{j_1} is regular. Similarly, u and w are compatible for any $w \in W \cap (Q(V) \setminus \{0\})$. By Theorem 2.4-13, v and w are compatible. Hence any two vectors of $((Q(V) \setminus \{0\}) \cap V_{j_1}) \cup ((Q(V) \setminus \{0\}) \cap W)$ are compatible. But $((Q(V) \setminus \{0\}) \cap V_{j_1}) \cup ((Q(V) \setminus \{0\}) \cap W)$ generates $V_{j_1} + W$. Hence $V_{j_1} + W$ contains a base B such that $B \subseteq ((Q(V) \setminus \{0\}) \cap V_{j_1}) \cup ((Q(V) \setminus \{0\}) \cap W)$. Therefore, by Theorem 2.4-16, $V_{j_1} + W$ is regular. Since W is maximal regular, $V_{j_1} + W = W$. Hence $V_{j_1} \subseteq W$ and $V_{j_0} + V_{j_1} \subseteq W$. For $u_k \in V_{j_k} \cap (Q(V) \setminus \{0\})$ $(k = 0, 1)$ there is a $\lambda \in F \setminus \{0\}$ such that $u_0 + \lambda u_1 \in Q(V) \setminus \{0\}$, since $u_0, u_1 \in W \cap (Q(V) \setminus \{0\})$. By (15) there exists a $j_2 \in J$ such that

$$u_0 + \lambda u_1 \in V_{j_2} \setminus \{0\}. \tag{16}$$

Since $u_0 + \lambda u_1 \notin V_{j_0}$ and $u_0 + \lambda u_1 \notin V_{j_1}$, $V_{j_2} \neq V_{j_0}$ and $V_{j_2} \neq V_{j_1}$. Hence, by the direct decomposition (14), $(V_{j_0} + V_{j_1}) \cap V_{j_2} = \{0\}$. This, however, contradicts (16). Consequently, every V_j in (14) is maximal and hence the theorem is proved.

We conclude this section with another theorem which results from Lemma 2.4-8.

Theorem 2.4-21. *Let V be a near vector space with quasi-kernel $Q(V)$. If $u \in Q(V) \setminus \{0\}$, $x \in V \setminus Fu$ and*

$$\alpha u + \beta x = \alpha' u + \beta' x \quad (\alpha, \beta, \alpha', \beta' \in F), \tag{17}$$

then $\alpha = \alpha'$ *and* $\beta = \beta'$.

Proof. Let $u =: u_0$. Extend $\{u\}$ to a base B of $Q(V)$ (see Theorem 1.3-9). By Theorem 2.4-8, there exists a linear combination $x = \sum_{i=0}^{r} \xi_i u_i$, with $\xi_i \in F$ and $u_i \in B$ $(0 \leq i \leq r)$. Since $x \notin Fu$, we can take $\xi_1 \neq 0$ without loss of generality. [If $\xi_i = 0$ for each i, with $1 \leq i \leq r$, then $x = \xi_0 u_0 \in Fu$.]

By (17), we have

$$\alpha u_0 + \beta \sum_{i=0}^{r} \xi_i u_i = \alpha' u_0 + \beta' \sum_{i=0}^{r} \xi_i u_i.$$

This implies that

$$(\alpha \underset{u_0}{+} \beta\xi_0) u_0 + \sum_{i=1}^{r} \beta\xi_i u_i = (\alpha' \underset{u_0}{+} \beta'\xi_0) u_0 + \sum_{i=1}^{r} \beta'\xi_i u_i.$$

Hence, as a result of the uniqueness of this representation (Theorem 2.4-8), we have

$$\alpha \underset{u_0}{+} \beta\xi_0 = \alpha' \underset{u_0}{+} \beta'\xi_0 \qquad \text{and} \qquad \beta\xi_1 = \beta'\xi_1.$$

Since $\xi_1 \neq 0$ we have, by (F_4), $\beta = \beta'$. Therefore, since $\alpha \underset{u_0}{+} \beta\xi_0 = \alpha' \underset{u_0}{+} \beta'\xi_0$, $\alpha = \alpha'$.

2.5 The structure of regular near vector spaces

Theorem 2.5-1. *A near vector space V is regular if and only if*

$$Q = \{\lambda v \mid \lambda \in F, v \in R_u\} =: FR_u, \tag{18}$$

where $R_u(V) = R_u$ is the kernel of a $u \in Q(V) \setminus \{0\} = Q \setminus \{0\}$. In this case $Q = FR_u$ for all $u \in Q \setminus \{0\}$.

Proof. Let V be a regular near vector space and let $u, v \in Q \setminus \{0\}$. Suppose that $v \in Fu$, i.e. $v = \lambda u$. Then $v \in FR_u$ since $u \in R_u$. Now, suppose that $v \notin Fu$. Then, since V is regular and $u, v \in Q \setminus \{0\}$, u *cp* v. Hence there exists a $\lambda \in F \setminus \{0\}$ such that $u + \lambda v \in Q$. Therefore, by Lemma 2.2-15, $\lambda v \in R_u$ since $\lambda v \notin Fu$. Hence $v \in FR_u$ $(\lambda^{-1}\lambda v \in FR_u)$. Therefore in both cases $Q \subseteq FR_u$. But, by Theorem 2.2-12, $FR_u \subseteq Q$. Hence (18) holds. Conversely, suppose that (18) holds for a $u \in Q \setminus \{0\}$. Then, for each $v \in Q \setminus \{0\}$, there exists a $v_0 \in R_u$ and an $\alpha \in F \setminus \{0\}$ such that $v = \alpha v_0$. Since $u, v_0 \in R_u$, it follows by Note 2.2-11, that $u + v_0 = u + \alpha^{-1} v \in R_u \subseteq Q$. Hence u *cp* v. But, since v is chosen arbitrarily, $Q \setminus \{0\}$ has only one equivalence class with respect to the relation *cp*. Hence V is regular.

Theorem 2.5-2 (The structure theorem for regular near vector spaces). *An F-group (V, F), with $V \neq \{0\}$, is a regular near vector space if and only if F is a near field and V is isomorphic to $F^{(I)}$, as defined in Theorem 2.4-3.*

Proof. Let $(F, +, \cdot)$ be a near field and I be a non empty index set. Then, by Theorem 2.4-3,

$$F^{(I)} := \{(\xi_i)_{i \in I} \mid \xi_i \in F, \xi_i \neq 0 \text{ for only a finite number of } i \in I\}$$

is a near vector space with addition and multiplication defined as follows:

$$(\xi_i) + (\eta_i) = (\xi_i + \eta_i)$$

and

$$\lambda(\xi_i) = (\lambda \xi_i) \qquad (\lambda, \xi_i, \eta_i \in F).$$

Now, suppose that V and $F^{(I)}$ are isomorphic. Then, without loss of generality, we can take V to be equal to $F^{(I)}$. By Definition 2.4-4, $K(F) = \{\kappa \in F \mid (\xi + \eta)\kappa = \xi\kappa + \eta\kappa$ for every $\xi, \eta \in F\}$ is the kernel of F. Moreover, by Theorem 2.4-7, $Q(F^{(I)}) = \{\lambda(\kappa_i)_{i \in I} \mid \lambda \in F$ and $\kappa_i \in K(F)\}$. Hence $Q(F^{(I)}) = FR$, where $R := \{(\kappa_i) \mid \kappa_i \in K\}$.
Now,

$$R_{e_j} = \{(\xi_i) \in F^{(I)} \mid (\alpha \underset{e_j}{+} \beta)(\xi_i) = \alpha(\xi_i) + \beta(\xi_i) \text{ for every } \alpha, \beta \in F\}$$

is the kernel of the linear F-group $(F^{(I)}, F)$ with respect to $e_j := (\delta_{ji})_{i \in I}$, where δ_{ji} is the Kronecker symbol. It can be shown as follows that $R_{e_j} = R$.
Let $(\kappa_i) \in R$. For R to be a subset of R_{e_j}, it suffices to show that $(\alpha \underset{e_j}{+} \beta)\ (\kappa_i) = \alpha(\kappa_i) + \beta(\kappa_i)$, for every $\alpha, \beta \in F$. First, we will show that $+ = \underset{e_j}{+}$. Let α and β be any elements of F. Then

$$\alpha e_j + \beta e_j = (\alpha + \beta)e_j.$$

Furthermore, by Definition 2.2-4, $\alpha e_j + \beta e_j = (\alpha \underset{e_j}{+} \beta)e_j$. Hence $\alpha + \beta = \alpha \underset{e_j}{+} \beta$.
Finally, let $\alpha, \beta \in F$ and $(\kappa_i) \in R$. Then

$$\begin{aligned}
\alpha(\kappa_i) + \beta(\kappa_i) &= (\alpha\kappa_i) + (\beta\kappa_i) \\
&= (\alpha\kappa_i + \beta\kappa_i) \\
&= ([\alpha + \beta]\kappa_i) \\
&= (\alpha + \beta)(\kappa_i) \\
&= (\alpha \underset{e_j}{+} \beta)(\kappa_i).
\end{aligned}$$

Hence $R \subseteq R_{e_j}$.
Now, let $(\xi_i) \in R_{e_j}$. Then, for every $\alpha, \beta \in F$,

$$(\alpha \underset{e_j}{+} \beta)(\xi_i) = \alpha(\xi_i) + \beta(\xi_i),$$

which implies that

$$(\alpha + \beta)(\xi_i) = \alpha(\xi_i) + \beta(\xi_i).$$

Therefore

$$\begin{aligned}([\alpha+\beta]\xi_i) &= (\alpha\xi_i)+(\beta\xi_i)\\ &= (\alpha\xi_i+\beta\xi_i).\end{aligned}$$

Hence, for each $i \in I$

$$(\alpha+\beta)\xi_i = \alpha\xi_i + \beta\xi_i.$$

This implies that $\xi_i \in K$ for each $i \in I$. Hence $(\xi_i) \in R$. Therefore $R_{e_j} \subseteq R$. Consequently $R_{e_j} = R$. Hence $Q = FR = FR_{e_j}$. Therefore, by (18), V is regular.

Conversely, let V be a regular near vector space. Then, by Note 2.4-2(a), V is a linear F-group.

Now, let $B = \{u_i \mid i \in I\}$ be a base of Q. Then, by Theorem 2.5-1, there exist $\lambda_i \in F \setminus \{0\}$ $(i \in I)$ such that $v_i := \lambda_i u_i \in R_{u_0}$ for a $u_0 \in B$. Hence, by Lemma 2.3-3, $B' := \{v_i \mid i \in I\} \subseteq R_{u_0}$ is a base of V. Moreover, by Theorem 2.2-6, $(F, \underset{u_0}{+}, \cdot)$ is a near field.

Define $f : V \to F^{(I)}$ by $f(x) := (\xi_i)_{i\in I}$. Then f is well defined. Let $x = \sum_{i\in I}\xi_i v_i$ and $y = \sum_{i\in I}\eta_i v_i$ be elements of V. Suppose that $x = y$. This implies that $\sum_{i\in I}\xi_i v_i = \sum_{i\in I}\eta_i v_i$. As a result of the uniqueness of the representation (Lemma 2.4-8), $\xi_i = \eta_i$ for each $i \in I$. Hence $(\xi_i) = (\eta_i)$ and so $f(x) = f(y)$.

Secondly, f respects the operations, i.e.

(i)

$$\begin{aligned} f(x+y) &= f(\sum_{i\in I}\xi_i v_i + \sum_{i\in I}\eta_i v_i)\\ &= f(\sum_{i\in I}[\xi_i \underset{u_0}{+} \eta_i]v_i)\\ &= (\xi_i \underset{u_0}{+} \eta_i)\\ &= (\xi_i) \underset{u_0}{+} (\eta_i)\\ &= f(x) \underset{u_0}{+} f(y),\end{aligned}$$

(ii)

$$\begin{aligned} \lambda f(x) &= \lambda f(\sum_{i\in I}\xi_i v_i)\\ &= \lambda(\xi_i)\\ &= f(\sum_{i\in I}\lambda\xi_i v_i)\\ &= f(\lambda\sum_{i\in I}\xi_i v_i)\\ &= f(\lambda x).\end{aligned}$$

Finally, we shall show that f is a bijection. Let $f(x) = f(y)$. Then

$$f(\sum_{i\in I}\xi_i v_i) = f(\sum_{i\in I}\eta_i v_i),$$

which implies that $(\xi_i) = (\eta_i)$. Hence $\xi_i = \eta_i$ for each $i \in I$. Therefore

$$\sum_{i\in I}\xi_i v_i = \sum_{i\in I}\eta_i v_i,$$

which implies that $x = y$. Hence f is injective. Furthermore, to show that f is surjective, let $(\xi_i)_{i\in I}$ be an element of $F^{(I)}$. Let $x = \sum_{i\in I}\xi_i v_i$. Then $x \in V$ since $\xi_i \neq 0$ for only a finite number of $i \in I$ and B' is a base of V. Hence $f(x) = f(\sum_{i\in I}\xi_i v_i) = (\xi_i)_{i\in I}$.

Theorem 2.5-3. *Let (V, F) be a near vector space with $dimV > 1$. Then F is a division ring and V a vector space over F if and only if $V = Q(V)$.*

Proof. Suppose that F is a division ring and V is a vector space over F. Let $\alpha, \beta \in F$ and $v \in V$. Then $\alpha v + \beta v = (\alpha + \beta)v$. Hence $v \in Q(V)$. Therefore $V = Q(V)$.

Conversely, suppose that $V = Q(V)$ and $dimV > 1$, i.e. $dimQ(V) > 1$. Then there exists a $u \in Q(V) \setminus \{0\}$. We shall show that $Q(V) = R_u$. If $v \in Q(V) \setminus Fu$, then by Lemma 2.2-15, $v \in R_u$. Therefore, suppose that $v \in Fu \setminus \{0\}$. Then, since $dimV > 1$, there exists a $w \in Q(V) \setminus Fu$. Hence, by Lemma 2.2-15, $w \in R_u$. But, since $v \notin Fw$ ($v = \lambda w$ implies that $w = \lambda^{-1}v \in Fu$), it follows, by Lemma 2.2-15, that $v \in R_u$. Hence $V = R_u = \{v \in V \mid (\alpha \underset{u}{+} \beta)v = \alpha v + \beta v\}$. Therefore $(\alpha \underset{u}{+} \beta)v = \alpha v + \beta v$ for each $v \in V$. Hence $\alpha \underset{u}{+} \beta = \alpha + \beta$. This implies that, for every $\alpha, \beta \in F$ and each $v \in V$,

$$(\alpha + \beta)v = \alpha v + \beta v. \tag{19}$$

Therefore, for each $x \in V$,

$$(\alpha + \beta)\gamma x = \alpha\gamma x + \beta\gamma x = (\alpha\gamma + \beta\gamma)x.$$

Hence, by (F_4),

$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma. \tag{20}$$

Therefore, by (20), F is a division ring and by (19) V is a vector space over F.

Note 2.5-4. (a) Theorem 2.5-3 does not hold when $dimV = 1$. This can be shown as follows. Let (V, F) be any near vector space of dimension one, i.e. $Q(V)$ is of dimension one. Let $\{v_0\}$ be a base of $Q(V)$ and let $v \in V$. Then $v = \lambda v_0$ for some $\lambda \in F$. Hence, by Lemma 2.2-2(c), $v \in Q(V)$. Hence $Q(V) = V$. Therefore if Theorem 2.5-3 holds for $dimV = 1$, the fact that $Q(V) = V$ would have implied that every near vector space of dimension one is a vector space. This contradicts Note 2.4-2(d).

(b) In Theorem 2.5-3 it is sufficient to require that (V, F) is an F-group. Indeed, if $V = Q(V)$, then $Q(V)$ generates V and hence, by definition, (V, F) is a near vector space.

Chapter 3

Examples of near vector spaces

3.1 Introduction

A vector space and a near field over itself are examples of near vector spaces which have already been given (see Note 2.4-2). In Section 3.2 we shall give three further examples of near vector spaces which are not vector spaces.

3.2 Examples

Each example (V, F) is given in five parts. First, we shall show that (V, F) is an F-group. Its quasi-kernel $Q(V)$ will then be investigated. Furthermore, we shall define $\underset{u}{+}$ on F for each $u \in Q(V) \setminus \{0\}$, after which the kernel R_u of (V, F) wil be determined for some $u \in Q(V) \setminus \{0\}$. Finally, we shall show that (V, F) is a near vector space and V will be decomposed into maximal regular near vector spaces.

Example 3.2-1. In this example we take $V := \mathbf{R}^2$, and we let a real number α act as an endomorphism of V by defining $\alpha(x_1, x_2) := (\alpha x_1, \alpha^3 x_2)$. In this way F is isomorphic to $\mathbf{R}$.

I It can be shown as follows that (V, F) is an F-group.

(F_1): $(V, +)$ is a group. Moreover, let $\alpha \in F$ and let $(x_1, x_2), (y_1, y_2) \in V$. Then

$$\begin{aligned} \alpha[(x_1, x_2) + (y_1, y_2)] &= \alpha(x_1 + y_1, x_2 + y_2) \\ &= (\alpha(x_1 + y_1), \alpha^3(x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha^3 x_2 + \alpha^3 y_2) \\ &= (\alpha x_1, \alpha^3 x_2) + (\alpha y_1, \alpha^3 y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2). \end{aligned}$$

Hence α is an endomorphism of V.

(F_2): Let $(x_1, x_2) \in V$. Then

$$0(x_1, x_2) = (0x_1, 0^3 x_2) = (0, 0),$$
$$1(x_1, x_2) = (1x_1, 1^3 x_2) = (x_1, x_2),$$

and

$$-1(x_1, x_2) = (-1x_1, (-1)^3 x_2) = (-x_1, -x_2).$$

(F_3): Let $(A, \cdot)$ be the automorphism group of $(V, +)$. We shall now show that $F^* \subseteq A$. Let $\alpha \in F^*$. Then α is an endomorphism. It suffices to show that α is a bijection.

Let $(x_1, x_2), (y_1, y_2) \in V$ and suppose that $\alpha(x_1, x_2) = \alpha(y_1, y_2)$. Then $(\alpha x_1, \alpha^3 x_2) = (\alpha y_1, \alpha^3 y_2)$, which implies that $\alpha x_1 = \alpha y_1$ and $\alpha^3 x_2 = \alpha^3 y_2$. Hence, since $\alpha \neq 0$ and F is a field, $x_1 = y_1$ and $x_2 = y_2$. Therefore α is injective. Furthermore, let $(x_1, x_2) \in V$. Then $(\alpha^{-1}x_1, \alpha^{-3}x_2) \in V$ and $\alpha(\alpha^{-1}x_1, \alpha^{-3}x_2) = (\alpha\alpha^{-1}x_1, \alpha^3\alpha^{-3}x_2) = (x_1, x_2)$. Hence α is surjective. Finally, since F is a field, $(F^*, \cdot)$ is a subgroup of $(A, \cdot)$.

(F_4): Let $(x_1, x_2) \in V$ and $\alpha, \beta \in F$. Suppose that $\alpha(x_1, x_2) = \beta(x_1, x_2)$. Then $(\alpha x_1, \alpha^3 x_2) = (\beta x_1, \beta^3 x_2)$, which implies that $\alpha x_1 = \beta x_1$ and $\alpha^3 x_2 = \beta^3 x_2$. Hence $\alpha = \beta$ or $x_1 = 0$ and $\alpha^3 = \beta^3$ or $x_2 = 0$. If $\alpha \neq \beta$, then $\alpha^3 \neq \beta^3$ and so $x_1 = 0$ and $x_2 = 0$. Hence $(x_1, x_2) = (0, 0)$.

II The quasi-kernel $Q(V)$ of V consists of all those elements u of V such that for every $\alpha, \beta \in F$ there exists a $\gamma \in F$ for which $\alpha u + \beta u = \gamma u$.

(i) Consider $(a, 0) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned}
\alpha(a,0) + \beta(a,0) &= (\alpha a, 0) + (\beta a, 0) \\
&= (\alpha a + \beta a, 0) \\
&= ([\alpha + \beta]a, 0) \\
&= [\alpha + \beta](a, 0).
\end{aligned}$$

Hence $(a, 0) \in Q(V)$ for each $a \in F$.

(ii) Consider $(0, b) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned}
\alpha(0,b) + \beta(0,b) &= (0, \alpha^3 b) + (0, \beta^3 b) \\
&= (0, \alpha^3 b + \beta^3 b) \\
&= (0, [\alpha^3 + \beta^3]b) \\
&= [\alpha^3 + \beta^3]^{\frac{1}{3}}(0, b).
\end{aligned}$$

Hence $(0, b) \in Q(V)$ for each $b \in F$.

Furthermore, consider $(a, b) \in V$ with $a \in F^*$ and $b \in F^*$. Then

$$\begin{aligned}
\alpha(a,b) + \beta(a,b) &= (\alpha a, \alpha^3 b) + (\beta a, \beta^3 b) \\
&= (\alpha a + \beta a, \alpha^3 b + \beta^3 b) \\
&= ([\alpha + \beta]a, [\alpha^3 + \beta^3]b) \\
&\neq \gamma(a, b),
\end{aligned}$$

if $(\alpha + \beta)^3 \neq \alpha^3 + \beta^3$. Hence

$$Q(V) = \{(a, 0) \mid a \in F\} \cup \{(0, b) \mid b \in F\}.$$

Therefore, since $Q(V) \neq \{(0,0)\}$, (V, F) is a linear F-group.

III For each $u \in Q(V) \setminus \{0\}$, define $\underset{u}{+}$ on F by $(\alpha \underset{u}{+} \beta)u := \alpha u + \beta u$.
(i) Let $u = (a, 0)$ with $a \in F^*$. Then

$$\alpha \underset{u}{+} \beta = \alpha + \beta.$$

(ii) Let $u = (0, b)$ with $b \in F^*$. Then

$$\alpha \underset{u}{+} \beta = (\alpha^3 + \beta^3)^{\frac{1}{3}}$$

We shall now show that, with addition as defined in (ii), $(F, \underset{u}{+}, \cdot)$ is a field (cf. Theorem 2.2-6). First, we shall show that $(F, \underset{u}{+})$ is an abelian group.
(a) Let α, β and γ be elements of F. Then

$$\begin{aligned}
(\alpha \underset{u}{+} \beta) \underset{u}{+} \gamma &= (\alpha^3 + \beta^3)^{\frac{1}{3}} \underset{u}{+} \gamma \\
&= [((\alpha^3 + \beta^3)^{\frac{1}{3}})^3 + \gamma^3]^{\frac{1}{3}} \\
&= [(\alpha^3 + \beta^3) + \gamma^3]^{\frac{1}{3}} \\
&= [\alpha^3 + (\beta^3 + \gamma^3)]^{\frac{1}{3}} \\
&= [\alpha^3 + ((\beta^3 + \gamma^3)^{\frac{1}{3}})^3]^{\frac{1}{3}} \\
&= \alpha \underset{u}{+} (\beta^3 + \gamma^3)^{\frac{1}{3}} \\
&= \alpha \underset{u}{+} (\beta \underset{u}{+} \gamma).
\end{aligned}$$

Hence $\underset{u}{+}$ is associative on F.
(b) The zero element of $(F, \underset{u}{+})$ is 0 since,

$$0 \underset{u}{+} \alpha = (0^3 + \alpha^3)^{\frac{1}{3}} = (\alpha^3)^{\frac{1}{3}} = \alpha$$

and

$$\alpha \underset{u}{+} 0 = (\alpha^3 + 0^3)^{\frac{1}{3}} = (\alpha^3)^{\frac{1}{3}} = \alpha.$$

(c) For each $\alpha \in F$,

$$\begin{aligned}
(-\alpha) \underset{u}{+} \alpha &= ((-\alpha)^3 + \alpha^3)^{\frac{1}{3}} \\
&= (-\alpha^3 + \alpha^3)^{\frac{1}{3}} \\
&= 0.
\end{aligned}$$

Similarly, $\alpha \underset{u}{+} (-\alpha) = 0$. Hence each $\alpha \in F$ has as inverse $-\alpha$.
(d) Let $\alpha, \beta \in F$. Then

$$\begin{aligned}
\alpha \underset{u}{+} \beta &= (\alpha^3 + \beta^3)^{\frac{1}{3}} \\
&= (\beta^3 + \alpha^3)^{\frac{1}{3}} \\
&= \beta \underset{u}{+} \alpha.
\end{aligned}$$

Hence $(F, \underset{u}{+})$ is an abelian group.

Secondly, let α, β and γ be elements of F. Then

$$\begin{aligned}
\alpha(\beta \underset{u}{+} \gamma) &= \alpha(\beta^3 + \gamma^3)^{\frac{1}{3}} \\
&= (\alpha^3)^{\frac{1}{3}}(\beta^3 + \gamma^3)^{\frac{1}{3}} \\
&= [\alpha^3(\beta^3 + \gamma^3)]^{\frac{1}{3}} \\
&= (\alpha^3\beta^3 + \alpha^3\gamma^3)^{\frac{1}{3}} \\
&= [(\alpha\beta)^3 + (\alpha\gamma)^3]^{\frac{1}{3}} \\
&= \alpha\beta \underset{u}{+} \alpha\gamma.
\end{aligned}$$

Similarly, $(\alpha \underset{u}{+} \beta)\gamma = \alpha\gamma \underset{u}{+} \beta\gamma$.

Finally, we know that $(F^*, \cdot)$ $(\approx (\mathbf{R}^*, \cdot))$ is an abelian group. Consequently $(F, \underset{u}{+}, \cdot)$ is a field.

By Theorem 2.2-8, we have that $(F, \underset{u}{+}, \cdot) \approx (F, \underset{\lambda u}{+}, \cdot)$. In this case, however, we shall show that $(F, \underset{u}{+}, \cdot) \approx (F, \underset{v}{+}, \cdot)$, where $\alpha \underset{u}{+} \beta = (\alpha^3 + \beta^3)^{\frac{1}{3}}$ and $\alpha \underset{v}{+} \beta = \alpha + \beta$. It suffices to show that $(F, \underset{u}{+}) \approx (F, \underset{v}{+})$. Define $f : F \to F$ by $f(x) = x^{\frac{1}{3}}$. Then, since $x = y$ implies that $x^{\frac{1}{3}} = y^{\frac{1}{3}}$, f is well defined.

Furthermore, it can be shown that f is a bijection. Let $f(x) = f(y)$, i.e $x^{\frac{1}{3}} = y^{\frac{1}{3}}$. Then

$$x = (x^{\frac{1}{3}})^3 = (y^{\frac{1}{3}})^3 = y.$$

Hence f is injective. Moreover, let $y \in F$. Then $x = y^3 \in F$ and $f(x) = f(y^3) = (y^3)^{\frac{1}{3}} = y$. Finally, f respects the operations:

$$\begin{aligned}
f(\alpha \underset{v}{+} \beta) &= (\alpha + \beta)^{\frac{1}{3}} \\
&= ((\alpha^{\frac{1}{3}})^3 + (\beta^{\frac{1}{3}})^3)^{\frac{1}{3}} \\
&= f(\alpha) \underset{u}{+} f(\beta).
\end{aligned}$$

IV The kernel $R_u(V)$ of V, with $u \in Q(V) \setminus \{0\}$, is defined by

$$R_u(V) := \{v \in V \mid (\alpha \underset{u}{+} \beta)v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\}.$$

Consider the following two cases:

(i) $(1, 0) \in Q(V) \setminus \{0\}$. Then $\alpha \underset{(1,0)}{+} \beta := \alpha + \beta$ and, for every $\alpha, \beta \in F$ and for each $a \in F$,

$$\alpha(a, 0) + \beta(a, 0) = (\alpha + \beta)(a, 0).$$

Hence $R_{(1,0)} = \{(a, 0) \mid a \in F\}$.

(ii) $(0, 1) \in Q(V) \setminus \{0\}$. Then $\alpha \underset{(0,1)}{+} \beta := (\alpha^3 + \beta^3)^{\frac{1}{3}}$ and, for every $\alpha, \beta \in F$ and for each $b \in F$,

$$\begin{aligned}\alpha(0,b)+\beta(0,b) &= (\alpha^3+\beta^3)^{\frac{1}{3}}(0,b) \\ &= (\alpha \underset{(0,1)}{+} \beta)(0,b).\end{aligned}$$

Hence $R_{(0,1)} = \{(0,b) \mid b \in F\}$.

V (V, F) is a near vector space, since (Q_1) holds: Let $(a,b) \in V$, then $(a,b) = a(1,0) + b^{\frac{1}{3}}(0,1)$, where $(1,0)$ and $(0,1) \in Q(V)$ and $a, b^{\frac{1}{3}} \in F^*$. Furthermore, $B = \{(1,0),(0,1)\}$ is a base of V. Hence V is a near vector space of dimension two. However, since (Q_2) does not hold $((a,0) + \lambda(0,b) = (a, \lambda^3 b) \notin Q(V))$, V is not regular and therefore not a vector space (see Note 1.2-1(c)).

We shall now show how V can be decomposed in maximal regular near vector spaces (cf. Theorem 2.4-17).

Let $Q^* := Q(V) \setminus \{0\}$. Then

$$Q^* = \{(a,0) \mid a \in F^*\} \cup \{(0,b) \mid b \in F^*\}.$$

Put

$$Q_1 := \{(a,0) \mid a \in F^*\} \quad \text{and} \quad Q_2 := \{(0,b) \mid b \in F^*\}.$$

Then

$$B_1 := B \cap Q_1 = \{(1,0)\} \quad \text{and} \quad B_2 := B \cap Q_2 = \{(0,1)\}.$$

Let $V_j :=< B_j >$, then

$$V_1 = \{(a,0) \mid a \in F\} \quad \text{and} \quad V_2 = \{(0,b) \mid b \in F\}.$$

It can be shown as follows that V_1 is a maximal regular near vector space. Let $(a_1,0)$ and $(a_2,0)$ be elements of $Q(V_1) \setminus \{0\}$ $(= Q_1)$ Then

$$(a_1,0) + (a_2,0) = (a_1 + a_2, 0) \in Q(V_1).$$

Hence V_1 is regular. Moreover, suppose that there exists a regular near vector space $W \supset V_1$ generated by $Q(W)$. Then there exists an $x \in Q(W) \backslash Q(V_1)$. Hence $x = (a,b)$ with $b \neq 0$. But (a,b) cp $(c,0)$ with $c \in F^*$. Therefore $(a,b) + \lambda(c,0) = (a + \lambda c, b) \in Q(V)$, which is a contradiction. Consequently V_1 is maximal regular. Similarly, V_2 is maximal regular. Furthermore, since $V_1 \cap V_2 = \{(0,0)\}$ and, for each $(a,b) \in V, (a,b) = (a,0) + (0,b), V = V_1 \dot{+} V_2$.

Let $I_j := B_j$ with $j = 1, 2$. Then

$$\begin{aligned}F^{(I_j)} :&= \{(\xi_i)_{i \in I_j} \mid \xi_i \in F\} \\ &= \{\xi \mid \xi \in F\} \\ &= F.\end{aligned}$$

Finally, it can be shown as follows that $V_j \approx F^{(I_j)}$ (cf. Theorem 2.5-2).

Define $g : V_2 \to F_2$, with $F_2 = (F, \underset{u}{+}, \cdot)$ where $\alpha \underset{u}{+} \beta = (\alpha^3 + \beta^3)^{\frac{1}{3}}$, by $g(0,b) = b^{\frac{1}{3}}$. Then g is well defined. Let $(0,b) = (0,c)$. Then $b = c$. Hence

$$g(0,b) = b^{\frac{1}{3}} = c^{\frac{1}{3}} = g(0,c).$$

Secondly, g is a bijection. Let $g(0,b) = g(0,c)$. Then $b^{\frac{1}{3}} = c^{\frac{1}{3}}$. Hence

$$b = (b^{\frac{1}{3}})^3 = (c^{\frac{1}{3}})^3 = c.$$

Therefore $(0,b) = (0,c)$ and so g is injective. Moreover, let $b \in F_2$. Then $(0,b^3) \in V_2$ and $g(0,b^3) = (b^3)^{\frac{1}{3}} = b$. Hence g is surjective.

Finally, we shall show that g respects the operations. Let $(0,b)$ and $(0,c)$ be elements of V_2. Then

$$\begin{aligned} g[(0,b) + (0,c)] &= g(0, b+c) \\ &= (b+c)^{\frac{1}{3}} \\ &= ((b^{\frac{1}{3}})^3 + (c^{\frac{1}{3}})^3)^{\frac{1}{3}} \\ &= b^{\frac{1}{3}} \underset{u}{+} c^{\frac{1}{3}} \\ &= g(0,b) \underset{u}{+} g(0,c) \end{aligned}$$

and

$$\begin{aligned} g[\lambda(0,b)] &= g(0, \lambda^3 b) \\ &= \lambda b^{\frac{1}{3}} \\ &= \lambda g(0,b). \end{aligned}$$

Similarly, $f : V_1 \to F^{(I_1)}$, defined by $f(a,0) = a$, is an isomorphism.

Example 3.2-2. In this example we take $V := (\mathbf{Z}_5)^4$, and we let $\alpha \in \mathbf{Z}_5$ act as an endomorphism of V by defining $\alpha(x_1, x_2, x_3, x_4) := (\alpha x_1, \alpha^3 x_2, \alpha^3 x_3, \alpha x_4)$. In this way F is isomorphic to $\mathbf{Z}_5$.

I It can be shown as follows that (V, F) is an F-group.

(F_1): $(V, +)$ is a group. Moreover, let $\alpha \in F$ and let (x_1, x_2, x_3, x_4), $(y_1, y_2, y_3, y_4) \in V$. Then

$$\begin{aligned} \alpha[(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)] &= \alpha(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \\ &= (\alpha(x_1 + y_1), \alpha^3(x_2 + y_2), \alpha^3(x_3 + y_3), \alpha(x_4 + y_4)) \\ &= (\alpha x_1 + \alpha y_1, \alpha^3 x_2 + \alpha^3 y_2, \alpha^3 x_3 + \alpha^3 y_3, \alpha x_4 + \alpha y_4) \\ &= (\alpha x_1, \alpha^3 x_2, \alpha^3 x_3, \alpha x_4) + (\alpha y_1, \alpha^3 y_2, \alpha^3 y_3, \alpha y_4) \\ &= \alpha(x_1, x_2, x_3, x_4) + \alpha(y_1, y_2, y_3, y_4). \end{aligned}$$

Hence α is an endomorphism of V.

(F_2): Let $(x_1, x_2, x_3, x_4) \in V$. Then

$$0(x_1, x_2, x_3, x_4) = (0x_1, 0^3x_2, 0^3x_3, 0x_4) = (0, 0, 0, 0),$$

$$1(x_1, x_2, x_3, x_4) = (1x_1, 1^3x_2, 1^3x_3, 1x_4) = (x_1, x_2, x_3, x_4),$$

and

$$\begin{aligned} -1(x_1, x_2, x_3, x_4) &= 4(x_1, x_2, x_3, x_4) \\ &= (4x_1, 4^3x_2, 4^3x_3, 4x_4) \\ &= (4x_1, 4x_2, 4x_3, 4x_4) \\ &= (-x_1, -x_2, -x_3, -x_4). \end{aligned}$$

(F_3): $F^* = \{1, 2, 3, 4\}$. Let $\alpha \in F^*$ and let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in V$.
First we shall show that α is a bijection.
(i) Suppose that $\alpha(x_1, x_2, x_3, x_4) = \alpha(y_1, y_2, y_3, y_4)$. Then

$$(\alpha x_1, \alpha^3 x_2, \alpha^3 x_3, \alpha x_4) = (\alpha y_1, \alpha^3 y_2, \alpha^3 y_3, \alpha y_4),$$

which implies that

$$\alpha x_1 = \alpha y_1, \alpha^3 x_2 = \alpha^3 y_2, \alpha^3 x_3 = \alpha^3 y_3 \quad \text{and} \quad \alpha x_4 = \alpha y_4.$$

Hence $\alpha(x_1 - y_1) = 0, \alpha^3(x_2 - y_2) = 0, \alpha^3(x_3 - y_3) = 0$ and $\alpha(x_4 - y_4) = 0$. Therefore, since $\alpha \neq 0, x_1 = y_1, x_2 = y_2, x_3 = y_3$ and $x_4 = y_4$. Hence $(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, y_4)$. Consequently α is injective.
(ii) Let $(x_1, x_2, x_3, x_4) \in V$ and let $\alpha \in F^*$. Then $(\alpha^{-1}x_1, \alpha^{-3}x_2, \alpha^{-3}x_3, \alpha^{-1}x_4) \in V$ and $\alpha(\alpha^{-1}x_1, \alpha^{-3}x_2, \alpha^{-3}x_3, \alpha^{-1}x_4) = (x_1, x_2, x_3, x_4)$. Hence α is surjective.
Furthermore, since α is an endomorphism and F is a field, F^* is a subgroup of the automorphism group of $(V, +)$.

(F_4): Let $(x_1, x_2, x_3, x_4) \in V$ and let $\alpha, \beta \in F$. Suppose that

$$\alpha(x_1, x_2, x_3, x_4) = \beta(x_1, x_2, x_3, x_4).$$

Then

$$(\alpha x_1, \alpha^3 x_2, \alpha^3 x_3, \alpha x_4) = (\beta x_1, \beta^3 x_2, \beta^3 x_3, \beta x_4),$$

which implies that $\alpha x_1 = \beta x_1$, $\alpha^3 x_2 = \beta^3 x_2$, $\alpha^3 x_3 = \beta^3 x_3$ and $\alpha x_4 = \beta x_4$. If $\alpha \neq \beta$, then $\alpha^3 \neq \beta^3$ and so $x_1 = x_2 = x_3 = x_4 = 0$, i.e. $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$.

II The quasi-kernel $Q(V)$ of V consists of all those elements u of V such that for every $\alpha, \beta \in F$ there exists a $\gamma \in F$ for which $\alpha u + \beta u = \gamma u$.
(i) Consider $(a, 0, 0, 0) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned}\alpha(a,0,0,0)+\beta(a,0,0,0) &= (\alpha a,0,0,0)+(\beta a,0,0,0)\\ &= (\alpha a+\beta a,0,0,0)\\ &= ((\alpha+\beta)a,0,0,0)\\ &= (\alpha+\beta)(a,0,0,0).\end{aligned}$$

Hence $(a,0,0,0) \in Q(V)$ for each $a \in F$.

(ii) Consider $(0,b,c,0) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned}\alpha(0,b,c,0)+\beta(0,b,c,0) &= (0,\alpha^3 b,\alpha^3 c,0)+(0,\beta^3 b,\beta^3 c,0)\\ &= (0,\alpha^3 b+\beta^3 b,\alpha^3 c+\beta^3 c,0)\\ &= (0,(\alpha^3+\beta^3)b,(\alpha^3+\beta^3)c,0)\\ &= (\alpha^3+\beta^3)^{\frac{1}{3}}(0,b,c,0).\end{aligned}$$

Hence $(0,b,c,0) \in Q(V)$ for every $b, c \in F$.

(iii) Consider $(0,0,0,d) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned}\alpha(0,0,0,d)+\beta(0,0,0,d) &= (0,0,0,\alpha d)+(0,0,0,\beta d)\\ &= (0,0,0,\alpha d+\beta d)\\ &= (0,0,0,(\alpha+\beta)d)\\ &= (\alpha+\beta)(0,0,0,d).\end{aligned}$$

Hence $(0,0,0,d) \in Q(V)$ for each $d \in F$.

It can be verified that elements of the form (a,b,c,d), $(a,b,c,0)$, $(a,b,0,0)$, $(a,0,0,d)$, $(a,0,c,0)$, $(0,b,0,d)$, $(0,0,c,d)$, $(a,b,0,d)$, $(0,b,c,d)$, $(a,0,c,d)$, with $a,b,c,d \in F^*$, are not elements of $Q(V)$. For example, $(a,b,0,0) \notin Q(V)$:

$$\begin{aligned}\alpha(a,b,0,0)+\beta(a,b,0,0) &= (\alpha a,\alpha^3 b,0,0)+(\beta a,\beta^3 b,0,0)\\ &= (\alpha a+\beta a,\alpha^3 b+\beta^3 b,0,0)\\ &= ((\alpha+\beta)a,(\alpha^3+\beta^3)b,0,0)\\ &\neq \gamma(a,b,0,0),\end{aligned}$$

if $\alpha^3+\beta^3 \neq (\alpha+\beta)^3$. Hence

$$Q(V) = \{(a,0,0,0) \mid a \in F\} \cup \{(0,b,c,0) \mid b,c \in F\} \cup \{(0,0,0,d) \mid d \in F\}.$$

Therefore, since $Q(V) \neq \{(0,0,0,0)\}$, (V,F) is a linear F-group.

III For each $u \in Q(V) \setminus \{0\}$, define $\underset{u}{+}$ on F by

$$(\alpha \underset{u}{+} \beta)u := \alpha u + \beta u.$$

(i) Let $u = (a,0,0,0)$ with $a \in F^*$. Then

$$\alpha \underset{u}{+} \beta = \alpha + \beta.$$

(ii) Let $u = (0, b, c, 0)$ with $b, c \in F$ and b and c not simultaneously zero. Then

$$\alpha \underset{u}{+} \beta = (\alpha^3 + \beta^3)^{\frac{1}{3}}.$$

(iii) Let $u = (0, 0, 0, d)$ with $d \in F^*$. Then

$$\alpha \underset{u}{+} \beta = \alpha + \beta.$$

IV The kernel $R_u(V)$ of V, with $u \in Q(V) \setminus \{0\}$, is defined by

$$R_u(V) := \{v \in V \mid (\alpha \underset{u}{+} \beta)v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\}.$$

Consider $(1, 0, 0, 0) \in Q(V) \setminus \{0\}$. Then $\alpha \underset{(1,0,0,0)}{+} \beta := \alpha + \beta$, and for every $\alpha, \beta \in F$ and $a \in F$,

$$\alpha(a, 0, 0, 0) + \beta(a, 0, 0, 0) = (\alpha + \beta)(a, 0, 0, 0).$$

Hence $R_{(1,0,0,0)} = \{(a, 0, 0, 0) \mid a \in F\}$. Similarly,

$$R_{(0,1,1,0)} = \{(0, b, c, 0) \mid b, c \in F\}$$

and

$$R_{(0,0,0,1)} = \{(0, 0, 0, d) \mid d \in F\}.$$

V Since (Q_1) holds, (V, F) is a near vector space: Let $(a, b, c, d) \in V$, then $(a, b, c, d) = a(1, 0, 0, 0) + b^{\frac{1}{3}}(0, 1, 0, 0) + c^{\frac{1}{3}}(0, 0, 1, 0) + d(0, 0, 0, 1)$, with $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ and $(0, 0, 0, 1)$ elements of $Q(V)$ and $a, b^{\frac{1}{3}}, c^{\frac{1}{3}}$ and d elements of F^*. Furthermore $B = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a base of V. Hence V is a near vector space of dimension four. However, since (Q_2) does not hold ($(a, 0, 0, 0) + \lambda(0, 0, 0, b) = (a, 0, 0, \lambda b) \notin Q(V)$), V is not regular and is therefore not a vector space.

We shall now show how V can be decomposed in maximal regular near vector spaces (cf. Theorem 2.4-17).

Let $Q^* := Q(V) \setminus \{0\}$. Then

$$Q^* = \{(a, 0, 0, 0) \mid a \in F^*\} \cup (\{(0, b, c, 0) \mid b, c \in F\} \setminus \{(0, 0, 0, 0)\}) \cup \{(0, 0, 0, d) \mid d \in F^*\}.$$

Put

$$Q_1 = \{(a, 0, 0, 0) \mid a \in F^*\},$$

$$Q_2 = \{(0, b, c, 0) \mid b, c \in F\} \setminus \{(0, 0, 0, 0)\}$$

and

$$Q_3 = \{(0,0,0,d) \mid d \in F^*\}.$$

Then

$$B_1 := B \cap Q_1 = \{(1,0,0,0)\},$$

$$B_2 := B \cap Q_2 = \{(0,1,0,0),(0,0,1,0)\}$$

and

$$B_3 := B \cap Q_2 = \{(0,0,0,1)\}.$$

Let $V_j =< B_j >$. Then

$$V_1 = \{(a,0,0,0) \mid a \in F\},$$

$$V_2 = \{(0,b,c,0) \mid b,c \in F\}$$

and

$$V_3 = \{(0,0,0,d) \mid d \in F\}.$$

It can be shown as follows that V_2 is a maximal regular near vector space. Every two elements of Q_2 are compatible. But $Q_2 = (Q \setminus \{0\}) \cap V_2$. Hence V_2 is regular. Moreover, suppose that there exists a regular near vector space $W \supset V_2$ generated by $Q(W)$. Then there exists $(a,b,c,d) \in Q(W) \setminus Q_2$ such that a or d is not zero. But W is regular and so (a,b,c,d) cp $(0,x_1,x_2,0)$ with $x_1, x_2 \in F^*$. Therefore $(a,b,c,d) + \lambda(0,x_1,x_2,0) = (a, b+\lambda^3 x_1, c+\lambda^3 x_2, d) \in Q(V)$, which is a contradiction. Consequently V_2 is maximal regular. Similarly, V_1 and V_3 are maximal regular. Furthermore, since $V_1 \cap V_2 \cap V_3 = \{(0,0,0,0)\}$ and, for each $(a,b,c,d) \in V$, $(a,b,c,d) = (a,0,0,0) + (0,b,c,0) + (0,0,0,d)$, $V = V_1 \dot{+} V_2 \dot{+} V_3$.

Let $I_j := B_j$ with $j = 1,2,3$. Then

$$\begin{aligned} F^{(I_1)} &:= \{(\xi_i)_{i \in I_1} \mid \xi_i \in F\} \\ &= \{\xi \mid \xi \in F\} \\ &= F, \end{aligned}$$

$$\begin{aligned} F^{(I_2)} &:= \{(\xi_i)_{i \in I_2} \mid \xi_i \in F\} \\ &= \{(\xi_1,\xi_2) \mid \xi_1, \xi_2 \in F\} \\ &= F^2 \end{aligned}$$

and

$$\begin{aligned} F^{(I_3)} &:= \{(\xi_i)_{i \in I_3} \mid \xi_i \in F\} \\ &= \{\xi \mid \xi \in F\} \\ &= F. \end{aligned}$$

Finally, it can be shown as follows that $V_j \approx F^{(I_j)}$ (cf. Theorem 2.5-2).

Define $f_1 : V_1 \to F^{(I_1)}$ by $f_1(a,0,0,0) = a$, $f_2 : V_2 \to F^{(I_2)}$ by $f_2(0,b,c,0) = (b,c)$, and $f_3 : V_3 \to F^{(I_3)}$ by $f_3(0,0,0,d) = d$. Then f_1, f_2 and f_3 are isomorphisms.

Example 3.2-3. Consider the field $(GF(3^2), +, \cdot)$ with

$$GF(3^2) := \{0, 1, 2, \gamma, 1+\gamma, 2+\gamma, 2\gamma, 1+2\gamma, 2+2\gamma\},$$

where γ is a zero of $x^9 - x$ and is not equal to $0, 1$ or 2. (The existence of $GF(p^n)$ is proved in [2]). The operations on $GF(3^2)$ can be defined as follows:

$$+ : (a + \gamma b) + (c + \gamma d) = (a + c)\, mod3 + \gamma(b + d)\, mod3$$

$\cdot$	0	1	2	γ	$1+\gamma$	$2+\gamma$	2γ	$1+2\gamma$	$2+2\gamma$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	γ	$1+\gamma$	$2+\gamma$	2γ	$1+2\gamma$	$2+2\gamma$
2	0	2	1	2γ	$2+2\gamma$	$1+2\gamma$	γ	$2+\gamma$	$1+\gamma$
γ	0	γ	2γ	2	$2+\gamma$	$2+2\gamma$	1	$1+\gamma$	$1+2\gamma$
$1+\gamma$	0	$1+\gamma$	$2+2\gamma$	$2+\gamma$	2γ	1	$2\gamma+1$	2	γ
$2+\gamma$	0	$2+\gamma$	$1+2\gamma$	$2+2\gamma$	1	γ	$\gamma+1$	2γ	2
2γ	0	2γ	γ	1	$2\gamma+1$	$\gamma+1$	2	$2\gamma+2$	$\gamma+2$
$1+2\gamma$	0	$1+2\gamma$	$2+\gamma$	$1+\gamma$	2	2γ	$2\gamma+2$	γ	1
$2+2\gamma$	0	$2+2\gamma$	$1+\gamma$	$1+2\gamma$	γ	2	$\gamma+2$	1	2γ

In [5], p.257, it is observed that $(GF(3^2), +, \circ)$, with

$$x \circ y := \begin{cases} x \cdot y & \text{if } x \text{ is a square in } (GF(3^2), +, \cdot) \\ x \cdot y^3 & \text{otherwise,} \end{cases}$$

is a near field but not a field.

$\circ$	0	1	2	γ	$1+\gamma$	$2+\gamma$	2γ	$1+2\gamma$	$2+2\gamma$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	γ	$1+\gamma$	$2+\gamma$	2γ	$1+2\gamma$	$2+2\gamma$
2	0	2	1	2γ	$2+2\gamma$	$1+2\gamma$	γ	$2+\gamma$	$1+\gamma$
γ	0	γ	2γ	2	$2+\gamma$	$2+2\gamma$	1	$1+\gamma$	$2\gamma+1$
$1+\gamma$	0	$1+\gamma$	$2+2\gamma$	$1+2\gamma$	2	γ	$2+\gamma$	2γ	1
$2+\gamma$	0	$2+\gamma$	$1+2\gamma$	$\gamma+1$	2γ	2	$2+2\gamma$	1	γ
2γ	0	2γ	γ	1	$2\gamma+1$	$\gamma+1$	2	$2\gamma+2$	$\gamma+2$
$1+2\gamma$	0	$1+2\gamma$	$2+\gamma$	$2\gamma+2$	γ	1	$1+\gamma$	2	2γ
$2+2\gamma$	0	$2+2\gamma$	$1+\gamma$	$\gamma+2$	1	2γ	$1+2\gamma$	γ	2

Define $\theta : GF(3^2) \to GF(3^2)$ by

$$\begin{aligned} \theta : \quad 0 &\to 0 \\ 1 &\to 1 \\ 2 &\to 2 \\ \gamma &\to 2\gamma \\ 2\gamma &\to \gamma \\ 1+\gamma &\to 1+\gamma \\ 2+2\gamma &\to 2+2\gamma \\ 2+\gamma &\to 1+2\gamma \\ 1+2\gamma &\to 2+\gamma. \end{aligned}$$

Then θ is an automorphism with respect to $\circ$, but $\theta(\alpha+\beta) \neq \theta(\alpha)+\theta(\beta)$, in general. If there is no danger of confusion $x \circ y$, with $x, y \in GF(3^2)$, is written as xy.

Let $V := GF(3^2)^2$ and let $\alpha \in GF(3^2)$ act as an endormorphism of V by defining $\alpha(a,b) := (\alpha a, \theta(\alpha)b)$. In this way F is isomorphic to $GF(3^2)$. Furthermore, let

$$A := \{1, 2, 1+\gamma, 2+2\gamma\}$$

and

$$B := \{\gamma, 2\gamma, 2+\gamma, 1+2\gamma\}$$

I It can be shown as follows that (V, F) is an F-group.

(F_1): $(V,+)$ is a group:

(i) $(0,0)$ is its zero element.

(ii) For each $(a+\gamma b, c+\gamma d) \in V$ there exists an inverse, namely,

$$-(a+\gamma b, c+\gamma d) = (-(a+\gamma b), \theta(-1)(c+\gamma d)) = (-(a+\gamma b), -(c+\gamma d)).$$

(iii) Assosiativity holds for $(V,+)$, since F is assosiative.

Moreover, let $\alpha \in F$ and let $(a,b),(c,d) \in V$. Then

$$\begin{aligned} \alpha[(a,b)+(c,d)] &= \alpha(a+c, b+d) \\ &= (\alpha(a+c), \theta(\alpha)(b+d)) \\ &= \begin{cases} (\alpha(a+c), \alpha(b+d)) & \text{if } \alpha \in A \\ (\alpha(a+c), -\alpha(b+d)) & \text{if } \alpha \in B \end{cases} \\ &= \begin{cases} (\alpha a+\alpha c, \alpha b+\alpha d) & \text{if } \alpha \in A \\ (\alpha a+\alpha c, -\alpha b-\alpha d) & \text{if } \alpha \in B \end{cases} \\ &= \begin{cases} (\alpha a, \alpha b)+(\alpha c, \alpha d) & \text{if } \alpha \in A \\ (\alpha a, -\alpha b)+(\alpha c, -\alpha d) & \text{if } \alpha \in B \end{cases} \\ &= (\alpha a, \theta(\alpha)b)+(\alpha c, \theta(\alpha)d) \\ &= \alpha(a,b)+\alpha(c,d). \end{aligned}$$

Hence α is an endomorphism of V.

(F_2): Let $(a,b) \in V$. Then

$$0(a,b) = (0a, \theta(0)b) = (0a, 0b) = (0,0),$$

$$1(a,b) = (1a, \theta(1)b) = (1a, 1b) = (a,b)$$

and

$$2(a,b) = (2a, \theta(2)b) = (2a, 2b) = (-a, -b).$$

(F_3): Let $\alpha \in F^*$. Since α is an endomorphism and F a near field, it suffices to show that α is a bijection.

(i) Let $\alpha(a,b) = \alpha(c,d)$ with (a,b) and $(c,d) \in V$.

Then

$$(\alpha a, \theta(\alpha)b) = (\alpha c, \theta(\alpha)d),$$

which implies that

$$(\alpha a, \alpha b) = (\alpha c, \alpha d) \quad \text{if} \quad \alpha \in A$$

and

$$(\alpha a, -\alpha b) = (\alpha c, -\alpha d) \quad \text{if} \quad \alpha \in B.$$

Hence, if $\alpha \in A$,

$$\alpha a = \alpha c \quad \text{and} \quad \alpha b = \alpha d$$

and, if $\alpha \in B$,

$$\alpha a = \alpha c \quad \text{and} \quad -\alpha b = -\alpha d.$$

Therefore, since $\alpha \neq 0$ and F does not contain any zero divisors,

$$a = c \quad \text{and} \quad b = d.$$

Hence

$$(a,b) = (c,d).$$

Consequently α is injective.

(ii) Let $(c,d) \in V$. Then, if $\alpha \in A$, $(\alpha^{-1}c, \alpha^{-1}d) \in V$ and $\alpha(\alpha^{-1}c, \alpha^{-1}d) = (c,d)$. If $\alpha \in B$, then $(\alpha^{-1}c, -\alpha^{-1}d) \in V$ and $\alpha(\alpha^{-1}c, -\alpha^{-1}d) = (c,d)$. Hence α is surjective.

(F_4): Let $\alpha, \beta \in F$ and $(a,b) \in V$. Suppose that

$$\alpha(a,b) = \beta(a,b).$$

Then

$$(\alpha a, \theta(\alpha)b) = (\beta a, \theta(\beta)b),$$

which implies that

$$\alpha a = \beta a \quad \text{and} \quad \theta(\alpha)b = \theta(\beta)b.$$

Let $\alpha \neq \beta$ and suppose that $a \neq 0$. Then there exists an $a^{-1} \in F$ such that $\alpha a a^{-1} = \beta a a^{-1}$, which implies that $\alpha = \beta$. This is a contradiction. Hence $a = 0$. Since θ is a bijection, $\alpha \neq \beta$ implies that $\theta(\alpha) \neq \theta(\beta)$. Now, suppose that $b \neq 0$. Then there exists a $b^{-1} \in F$ such that $\theta(\alpha)bb^{-1} = \theta(\beta)bb^{-1}$, which implies that $\theta(\alpha) = \theta(\beta)$. This is a contradiction. Hence $b = 0$. Therefore

$$(a, b) = (0, 0).$$

II The quasi-kernel $Q(V)$ of V consists of all those elements u of V such that for every $\alpha, \beta \in F$ there exists a $\gamma \in F$ for which $\alpha u + \beta u = \gamma u$.

(i) Consider $(1, 0) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned} \alpha(1,0) + \beta(1,0) &= (\alpha, 0) + (\beta, 0) \\ &= (\alpha + \beta, 0) \\ &= (\alpha + \beta)(1, 0). \end{aligned}$$

Hence $(1, 0) \in Q(V)$.

Now, consider $(c, 0) \in V$ with $c \in F^*$. Then, since $(1, 0) \in Q(V)$,

$$\begin{aligned} \alpha(c,0) + \beta(c,0) &= \alpha c(1,0) + \beta c(1,0) \\ &= (\alpha c + \beta c)(1, 0) \\ &= (\alpha c + \beta c)c^{-1}(c, 0). \end{aligned}$$

Hence $(c, 0) \in Q(V)$ for each $c \in F^*$.

(ii) Consider $(0, 1) \in V$. For $\alpha, \beta \in F$,

$$\begin{aligned} \alpha(0,1) + \beta(0,1) &= (0, \theta(\alpha)) + (0, \theta(\beta)) \\ &= (0, \theta(\alpha) + \theta(\beta)) \\ &= \theta^{-1}(\theta(\alpha) + \theta(\beta))(0, 1). \end{aligned}$$

Since $\theta : F \to F$ is a bijection, $\theta^{-1} : F \to F$ exists. Hence $\theta^{-1}(\theta(\alpha) + \theta(\beta)) \in F$ and therefore $(0, 1) \in Q(V)$.

Now, consider $(0, c) \in V$ with $c \in F^*$. Then, for $\alpha, \beta \in F$,

$$\begin{aligned} \alpha(0,c) + \beta(0,c) &= \alpha\theta^{-1}(c)(0,1) + \beta\theta^{-1}(c)(0,1) \\ &= \theta^{-1}(\theta(\alpha\theta^{-1}(c)) + \theta(\beta\theta^{-1}(c)))(0,1) \\ &= \theta^{-1}(\theta(\alpha)c + \theta(\beta)c)(0,1) \\ &= \theta^{-1}(\theta(\alpha)c + \theta(\beta)c)\theta^{-1}(c^{-1})(0,c) \\ &= \theta^{-1}((\theta(\alpha)c + \theta(\beta)c)c^{-1})(0,c). \end{aligned}$$

Hence $(0, c) \in Q(V)$ for each $c \in F^*$.

Furthermore, consider $(a, b) \in V$ with $a, b \in F^*$. Then, for $\alpha, \beta \in F$,

$$\begin{aligned}\alpha(a,b) + \beta(a,b) &= (\alpha a, \theta(\alpha)b) + (\beta a, \theta(\beta)b) \\ &= (\alpha a + \beta a, \theta(\alpha)b + \theta(\beta)b) \\ &\neq \gamma(a,b),\end{aligned}$$

if $\theta^{-1}((\theta(\alpha)b + \theta(\beta)b)b^{-1}) \neq (\alpha a + \beta a)a^{-1}$. Hence $(a, b) \notin Q(V)$ if $a, b \in F^*$. Therefore

$$Q(V) = \{(c, 0) \mid c \in F\} \cup \{(0, c) \mid c \in F\}.$$

Hence, since $Q(V) \neq \{(0,0)\}, (V, F)$ is a linear F-group.

III For each $u \in Q(V) \setminus \{0\}$, define $\underset{u}{+}$ on F by $(\alpha \underset{u}{+} \beta) := \alpha u + \beta u$.
(i) Let $u = (c, 0)$ with $c \in F^*$. Then

$$\alpha \underset{u}{+} \beta = (\alpha c + \beta c)c^{-1}.$$

(ii) Let $u = (0, c)$ with $c \in F^*$. Then

$$\alpha \underset{u}{+} \beta = \theta^{-1}((\theta(\alpha)c + \theta(\beta)c)c^{-1}).$$

By Theorem 2.2-6 $(F, \underset{u}{+}, \circ)$ is a near field for each $u \in Q(V) \setminus \{0\}$. However, $(F, \underset{u}{+}, \circ)$, in general, is not a field: For example, consider $(F, \underset{(\gamma,0)}{+}, \circ)$. Let $\alpha = 2 + \gamma, \beta = 2\gamma$ and $\xi = 1$. Then

$$\begin{aligned}(\beta \underset{u}{+} \xi) \circ \alpha &= (2\gamma \underset{u}{+} 1) \circ (2 + \gamma) \\ &= ((2\gamma \circ \gamma + 1 \circ \gamma) \circ 2\gamma) \circ (2 + \gamma) \\ &= ((1 + \gamma) \circ 2\gamma) \circ (2 + \gamma) \\ &= 2\end{aligned}$$

and

$$\begin{aligned}\beta \circ \alpha \underset{u}{+} \xi \circ \alpha &= (2\gamma \circ (2 + \gamma)) \underset{u}{+} (1 \circ (2 + \gamma)) \\ &= (\gamma + 1) \underset{u}{+} (2 + \gamma) \\ &= ((\gamma + 1) \circ \gamma + (2 + \gamma) \circ \gamma) \circ 2\gamma \\ &= 2 \circ 2\gamma \\ &= \gamma.\end{aligned}$$

But $2 \neq \gamma$. Hence $(\beta \underset{u}{+} \xi) \circ \alpha \neq \beta \circ \alpha \underset{u}{+} \xi \circ \alpha$.

IV The kernel $R_u(V)$ of V, with $u \in Q(V) \setminus \{0\}$, is defined by

$$R_u(V) := \{v \in V \mid (\alpha \underset{u}{+} \beta)v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\}.$$

(i)

$$\begin{aligned} R_{(1,0)} &= \{v \in V \mid (\alpha+\beta)v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\} \\ &= \{(0,0),(1,0),(2,0)\}. \end{aligned}$$

(ii)

$$\begin{aligned} R_{(0,1)} &= \{v \in V \mid \theta^{-1}(\theta(\alpha)+\theta(\beta))v = \alpha v + \beta v \quad \text{for} \quad \text{every} \quad \alpha, \beta \in F\} \\ &= \{(0,0),(0,1),(0,2)\}. \end{aligned}$$

V (V,F) is a near vector space, since (Q_1) holds: Let $(a,b) \in V$, then $(a,b) = a(1,0) + \theta^{-1}(b)(0,1)$, with $(1,0)$ and $(0,1)$ elements of $Q(V)$. However, since (Q_2) does not hold $((0,1)+\lambda(1,0) = (\lambda,1) \notin Q(V)), (V,F)$ is not a regular near vector space. Therefore V is a near vector space, but not a vector space, over F.

We shall now show how V can be decomposed in maximal regular near vector spaces (cf. Theorem 2.4-17).

Let $Q^* := Q(V) \setminus \{0\}$. Then

$$Q^* = \{(a,0) \mid a \in F^*\} \cup \{(0,a) \mid a \in F^*\}.$$

Put

$$Q_1 := \{(a,0) \mid a \in F^*\}$$

and

$$Q_2 := \{(0,a) \mid a \in F^*\}.$$

But $B = \{(1,0),(0,1)\}$ is a base of $Q(V)$. Hence $B_1 := B \cap Q_1 = \{(1,0)\}$ and $B_2 := B \cap Q_2 = \{(0,1)\}$. Let $V_j =< B_j >$. Then

$$V_1 = \{(a,0) \mid a \in F\}.$$

and

$$V_2 = \{(0,a) \mid a \in F\}.$$

Since $V_1 \cap V_2 = \{(0,0)\}$ and, for each $(a,b) \in V$, $(a,b) = (a,0) + (0,b), V = V_1 \dot{+} V_2$. Moreover, it can be shown as follows that $V_j (j = 1,2)$ are maximal regular. Consider V_1. Then since $Q_1 = (Q(V) \setminus \{0\}) \cap V_1$ and every two elements in Q_1 are compatible, V_1 is regular. Furthermore, suppose that there exists a regular near vector space $W \supset V_1$ generated by $Q(W)$. Then there exists $(a,b) \in Q(W) \setminus Q_1$ such that $b \neq 0$. But, since W is regular, $(a,b)\, cp\, (c,0)$ with $c \in F^*$. Hence $(a,b) + \lambda(c,0) = (a + \lambda c, b) \in Q(V)$, which is a contradiction. Consequently V_1 is maximal regular. Similarly V_2 is maximal regular.

(V_2, F_2), with $F_2 := (F, \underset{(0,1)}{+}, \circ)$ is a regular near vector space. Hence, by Theorem 2.5-2, $V_2 \approx F_2^{(I_2)}$ for some index set I_2. Let $I_2 = B_2$. Then $g : V_2 \to F_2^{(I_2)}$, defined by $g(0,b) = \theta^{-1}(b)$, is an isomorphism. This can be shown as follows. Suppose that $(0,b) = (0,c)$. Then $b = c$. Hence, since θ is a bijection, $\theta^{-1}(b) = \theta^{-1}(c)$. This implies that $g(0,b) = g(0,c)$. Therefore g is well defined.

Furthermore, g is an injection. Suppose that $g(0,b) = g(0,c)$. Then $\theta^{-1}(b) = \theta^{-1}(c)$. But θ^{-1} is injective. Hence $b = c$. This implies that $(0,b) = (0,c)$. Moreover, let $b \in F_2$. Then $(0,\theta(b)) \in V_2$ and $g(0,\theta(b)) = \theta^{-1}\theta(b) = b$.

Finally, we shall show that g respects the operations. Let $(0,b)$ and $(0,c)$ be elements of V_2. Then

$$\begin{aligned} g((0,b)+(0,c)) &= g(0,b+c)) \\ &= \theta^{-1}(b+c) \\ &= \theta^{-1}(\theta(\theta^{-1}(b)) + \theta(\theta^{-1}(c))) \\ &= \theta^{-1}(b) \underset{(0,1)}{+} \theta^{-1}(c) \\ &= g(0,b) \underset{(0,1)}{+} g(0,c) \end{aligned}$$

and

$$\begin{aligned} g(\lambda(0,b)) &= g(0,\theta(\lambda)b) \\ &= \theta^{-1}(\theta(\lambda)b) \\ &= \lambda\theta^{-1}(b) \\ &= \lambda g(0,b). \end{aligned}$$

Similarly, $V_1 \approx F_1^{(I_1)}$ where $F_1 := (F, \underset{(1,0)}{+}, \circ)$ and $I_1 = B_1$. The function $g' : V_1 \to F_1^{(I_1)}$, defined by $g'(a,0) = a$, is an isomorphism.

Chapter 4

Near linear transformations

4.1 Introduction

In Section 4.2 we recall some well-known facts about vector spaces. These ideas will then, in Section 4.3, be investigated for near vector spaces.

4.2 Linear transformations of certain vector spaces

Let V be a vector space of dimension two over a field F, i.e. $V \approx F^2$, with $\alpha\binom{x}{y} := \binom{\alpha x}{\alpha y}$. Furthermore, let

$$N := M_F(V) = \{f : V \to V \mid f(\sigma u) = \sigma f(u) \text{ for each } \sigma \in F \text{ and each } u \in V\}.$$

Then it can be shown as follows that N is a 2-primitive near ring on V. Firstly, we know that $M_F(V)$ is a near ring. Moreover, for $u \neq 0$ and for each $x \in V$, there exists a $f_{u,x} \in M_F(V)$ defined by

$$f_{u,x}(y) = \begin{cases} \alpha x & \text{if } y \neq 0 \text{ and } y = \alpha u \text{ for some } \alpha \in F, \\ 0 & \text{otherwise.} \end{cases}$$

(a) $f_{u,x}$ is well defined:

(i) Let $y \neq 0$ and $y = \alpha u$. Suppose that $\alpha u = \alpha' u$. Then $(\alpha - \alpha')u = 0$. Therefore, since $u \neq 0$, $\alpha = \alpha'$.

(ii) Let $y \neq 0$ and $y \neq \alpha u$ for each $\alpha \in F$. Then $f_{u,x}(y) = 0$.

(iii) Let $y = 0$. Then $f_{u,x}(y) = 0$.

(b) $f_{u,x} \in N$:

It suffices to show that, for each $\beta \in F$,

$$f_{u,x}(\beta y) = \beta f_{u,x}(y).$$

(i) Let $y = 0$. Then, for each $\beta \in F$, $\beta y = 0$. Hence $f_{u,x}(\beta y) = 0$. But $f_{u,x}(y) = 0$. Therefore $\beta f_{u,x}(y) = 0$.

(ii) Let $y \neq 0$ and $\beta = 0$. Then $f_{u,x}(\beta y) = f_{u,x}(0) = 0$ and $\beta f_{u,x}(y) = 0 f_{u,x}(y) = 0$.

(iii) Let $y \neq 0$, $\beta \neq 0$ and $\beta y = \alpha u$ for some $\alpha \in F$. Then $y = \beta^{-1}\alpha u$. Hence $f_{u,x}(\beta y) = \alpha x$ and $f_{u,x}(y) = \beta^{-1}\alpha x$. Therefore $f_{u,x}(\beta y) = \beta f_{u,x}(y)$.

(iv) Let $y \neq 0, \beta \neq 0$ and $\beta y \neq \alpha u$ for each $\alpha \in F$. Then $y \neq \alpha u$ for each $\alpha \in F$. Hence $f_{u,x}(y) = 0 = f_{u,x}(\beta y)$.

It can now be shown that V is monogenic, i.e. there exists a $u \neq 0$ such that $M_F(V)u = V$. Take any $u \neq 0$ and any $x \in V$, then, since $u = 1u$, $f_{u,x}(u) = x$. Also, V is of type 2.

Suppose that H is an $M_F(V)$-subgroup of V with $H \neq \{0\}$ and $H \neq V$. Take any $h \neq 0$ in H and any $x \in V \setminus H$. Then

$$f_{h,x}(h) = x \notin H,$$

which is a contradiction. Hence $H = \{0\}$ or $H = V$.

Finally, since $f_{u,x}(u) = 0$ for each $u \in V$ implies that $f_{u,x} = 0$, V is faithful.

Furthermore, since $V \approx F^2$, $\{\binom{1}{0}, \binom{0}{1}\}$ is a base of V. If T is a linear transformation of V, i.e.

$$T(v+u) = Tv + Tu$$

and

$$T(\alpha u) = \alpha T(u),$$

then T can be represented by a matrix $\binom{a\ b}{c\ d}$ where $T\binom{1}{0} = \binom{a}{c}$ and $T\binom{0}{1} = \binom{b}{d}$.

Now, consider the matrix ring $\mathbf{M}_2(F)$ over F as a near ring. Then it can be shown as follows that $\mathbf{M}_2(F)$ is 2-primitive on V. First, we show that each non-zero element of V is a generator. Suppose that $\binom{x}{y} \neq \binom{0}{0}$ and let $\binom{u}{v}$ be any element of V. We have to find an element f_m of $\mathbf{M}_2(F)$ such that

$$f_m\binom{x}{y} = \binom{u}{v},$$

that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y} = \binom{u}{v}.$$

Therefore a, b, c and d have to satisfy $ax + by = u$ and $cx + dy = v$. Now, if $y \neq 0$, put $a = 1$ and $c = 0$. Then $b = (u-x)y^{-1}$ and $d = vy^{-1}$. Hence $\begin{pmatrix} 1 & (u-x)y^{-1} \\ 0 & vy^{-1} \end{pmatrix}$ is an element of $\mathbf{M}_2(F)$ such that

$$\begin{pmatrix} 1 & (u-x)y^{-1} \\ 0 & vy^{-1} \end{pmatrix}\binom{x}{y} = \binom{u}{v}.$$

If $y = 0$, then $x \neq 0$. Therefore put $a = ux^{-1}$ and $c = vx^{-1}$. Then $\begin{pmatrix} ux^{-1} & b \\ vx^{-1} & d \end{pmatrix}$ is an element of $\mathbf{M}_2(F)$ such that

$$\begin{pmatrix} ux^{-1} & b \\ vx^{-1} & d \end{pmatrix}\binom{x}{0} = \binom{u}{v}.$$

Therefore V is monogenic. Moreover, since $\mathbf{M}_2(F)\binom{x}{y} = V$ for each $\binom{x}{y} \in V \setminus \{0\}$, V does not contain any proper submodules. Hence V is of type 2.

Finally, since $\mathbf{M}_2(F)$ is a set of functions on V, V is faithful.

Now, $\mathbf{M}_2(F) \subseteq N$. However, it can be shown as follows that $\mathbf{M}_2(F)$ is a proper subset of N. Let $\binom{1}{q} \in V$. Define $f_q : V \to V$ by

$$f_q\binom{x}{y} = \begin{cases} \binom{x}{y} & \text{if } x \neq 0 \text{ and } yx^{-1} = q \\ 0 & \text{otherwise.} \end{cases}$$

First, we shall show that f_q is an element of N. Let $\alpha \in F$ and $\binom{x}{y} \in V$ with $x \neq 0$. Then

$$f_q(\alpha\binom{x}{y}) = f_q\binom{\alpha x}{\alpha y} = \begin{cases} \binom{\alpha x}{\alpha y} & \text{if } \alpha y(\alpha x)^{-1} = q \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha(f_q\binom{x}{y}) = \begin{cases} \alpha\binom{x}{y} & \text{if } yx^{-1} = q \\ 0 & \text{otherwise.} \end{cases}$$

Hence $f_q(\alpha\binom{x}{y}) = \alpha(f_q\binom{x}{y})$.

Finally, we shall show, by means of a contradiction, that $f_q \notin \mathbf{M}_2(F)$. Take $q = 0$ and suppose that f_0 can be represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $(a, b, c, d \in F)$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{1}{0} = \binom{a}{c} = \binom{1}{0}.$$

Hence $a = 1$ and $c = 0$. Moreover,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{0}{1} = \binom{b}{d} = \binom{0}{0}.$$

Hence $b = 0$ and $d = 0$. Therefore $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. But

$$f_0\binom{2}{1} = \binom{0}{0}.$$

However,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\binom{2}{1} = \binom{2}{0}.$$

Hence $f_0 \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which is a contradiction.

4.3 Near linear transformations of certain near vector spaces

Definition 4.3-1. Let (V, F) be a near vector space. A function $g : V \to V$ is called a *near linear transformation* of V if for each $x \in V$ and for each $\alpha \in F$,

$$g(\alpha x) = \alpha g(x).$$

Consider the near vector space (V, F) as defined in Example 3.2-1, i.e. $V := \mathbf{R}^2$, $F \approx \mathbf{R}$ and $\alpha\binom{x}{y} := \binom{\alpha x}{\alpha^3 y}$ with $\alpha \in \mathbf{R}$ and $\binom{x}{y} \in V$. Furthermore, let

$$N := \{f : V \to V \mid f \text{ is a near linear transformation of} V\}.$$

Then N is a 2-primitive near ring on V. The proof of this is similar to the proof of the 2-primitivity of N in Section 4.2.

Let $\mathbf{M}_2(F)$ be the 2×2 matrix ring over F. For each $A = \binom{a_{11}\ a_{12}}{a_{21}\ a_{22}} \in \mathbf{M}_2(F)$, we define the mapping $A_\phi : V \to V$ by

$$A_\phi\binom{x_1}{x_2} = \binom{a_{11}x_1 + a_{12}x_2^{\frac{1}{3}}}{a_{21}x_1^3 + a_{22}x_2}.$$

The mapping $A_\phi : V \to V$ is an element of N. This can be shown as follows. Let $\binom{x_1}{x_2} \in V$ and $\alpha \in F$. Then

$$\begin{aligned} A_\phi(\alpha\binom{x_1}{x_2}) &= A_\phi(\binom{\alpha x_1}{\alpha^3 x_2}) \\ &= \binom{a_{11}\alpha x_1 + a_{12}\alpha x_2^{\frac{1}{3}}}{a_{21}\alpha^3 x_1^3 + a_{22}\alpha^3 x_2} \\ &= \binom{\alpha(a_{11}x_1 + a_{12}x_2^{\frac{1}{3}})}{\alpha^3(a_{21}x_1^3 + a_{22}x_2)} \\ &= \alpha\binom{a_{11}x_1 + a_{12}x_2^{\frac{1}{3}}}{a_{21}x_1^3 + a_{22}x_2} \\ &= \alpha(A_\phi\binom{x_1}{x_2}). \end{aligned}$$

However,

$$\begin{aligned} A_\phi(\binom{x_1}{x_2} + \binom{y_1}{y_2}) &= A_\phi(\binom{x_1 + y_1}{x_2 + y_2}) \\ &= \binom{a_{11}(x_1 + y_1) + a_{12}(x_2 + y_2)^{\frac{1}{3}}}{a_{21}(x_1 + y_1)^3 + a_{22}(x_2 + y_2)} \end{aligned}$$

and

$$\begin{aligned} A_\phi\binom{x_1}{x_2} + A_\phi\binom{y_1}{y_2} &= \binom{a_{11}x_1 + a_{12}x_2^{\frac{1}{3}}}{a_{21}x_1^3 + a_{22}x_2} + \binom{a_{11}y_1 + a_{12}y_2^{\frac{1}{3}}}{a_{21}y_1^3 + a_{22}y_2} \\ &= \binom{a_{11}(x_1 + y_1) + a_{12}(x_2^{\frac{1}{3}} + y_2^{\frac{1}{3}})}{a_{21}(x_1^3 + y_1^3) + a_{22}(x_2 + y_2)}. \end{aligned}$$

Hence, in general, $A_\phi(\binom{x_1}{x_2} + \binom{y_1}{y_2}) \neq A_\phi\binom{x_1}{x_2} + A_\phi\binom{y_1}{y_2}$. Therefore $A_\phi : V \to V$ is not a linear transformation.

Let $S := \{A_\phi : A \in \mathbf{M}_2(F)\}$. Then $S \subseteq N$. Furthermore, let T be the near ring generated by S, i.e. the intersection of all the sub near rings of N which contain S. It can be shown as follows that T is 2-primitive on V. First, we know that T is a sub near ring of N. Hence, since V is an N-module, V is a T-module. But T is a set of functions on V, hence V is a faithful T-module.

Secondly, V is monogenic. Let $\binom{x_1}{x_2} \in V$. Then $\binom{x_1\ x_3}{x_2\ x_4}_\phi \in T$ and $\binom{x_1\ x_3}{x_2\ x_4}_\phi\binom{1}{0} = \binom{x_1+0}{x_2+0} = \binom{x_1}{x_2}$. Hence $T\binom{1}{0} = V$.

Finally, V is of type 2. Let H be a T-submodule of V and let $\binom{x_1}{x_2} \in H$ with $\binom{x_1}{x_2} \neq \binom{0}{0}$. If $x_1 \neq 0$, then, since $TH \subseteq H$,

$$\begin{pmatrix} x_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_\phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H.$$

Hence $TH = V$. Therefore $H = V$. If $x_1 = 0$, then

$$\begin{pmatrix} 0 & x_2^{-\frac{1}{3}} \\ 0 & 0 \end{pmatrix}_\phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H,$$

and so, as before, $H = V$. Consequently V does not contain any proper non-trivial T-submodules.

By the definition of T, T is a sub near ring of N. We shall now show that T is a proper sub near ring of N. Consider the orbit $B := F\binom{1}{1} = \{\binom{\alpha}{\alpha^3} \mid \alpha \in F\}$. Define $e : V \to V$ by

$$e\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \binom{x}{y} & \text{if } \binom{x}{y} \in B \\ 0 & \text{if } \binom{x}{y} \notin B. \end{cases}$$

Then

$$\begin{aligned} e(\alpha\begin{pmatrix} x \\ y \end{pmatrix}) &= e\begin{pmatrix} \alpha x \\ \alpha^3 y \end{pmatrix} \\ &= \begin{pmatrix} \alpha x \\ \alpha^3 y \end{pmatrix} \\ &= \alpha\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \alpha(e\begin{pmatrix} x \\ y \end{pmatrix}), \end{aligned}$$

if $\binom{x}{y} \in B$, and

$$\begin{aligned} e(\alpha\begin{pmatrix} x \\ y \end{pmatrix}) &= e\begin{pmatrix} \alpha x \\ \alpha^3 y \end{pmatrix} \\ &= 0 \\ &= \alpha e\begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

if $\binom{x}{y} \notin B$. Hence $e \in N$.

Now, elements of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\phi$ and products and sums of such elements are the only elements of T. We shall now show that all the elements of T are continuous. To do this, we need the next lemma.

Lemma 4.3-2. *Let $f(x,y) = f_1(x) + f_2(y)$, with $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ continuous on $\mathbf{R}$. Then $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous on $\mathbf{R}^2$.*

Proof. Let $(x_0, y_0) \in \mathbf{R}^2$. Then f_1 is continuous at x_0 and f_2 is continuous at y_0.

Let $\varepsilon > 0$. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that,

$$| f_1(x) - f_1(x_0) | < \frac{\varepsilon}{2} \quad \text{if} \quad | x - x_0 | < \delta_1$$

and

$$| f_2(y) - f_2(y_0) | < \frac{\varepsilon}{2} \quad \text{if} \quad | y - y_0 | < \delta_2.$$

Let $\delta = min\{\delta_1, \delta_2\}$. If

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

then

$$| x - x_0 | < \delta \leq \delta_1 \quad \text{and} \quad | y - y_0 | < \delta \leq \delta_2.$$

Therefore

$$\begin{aligned} | f(x, y) - f(x_0, y_0) | &= | f_1(x) + f_2(y) - (f_1(x_0) + f_2(y_0)) | \\ &= | f_1(x) - f_1(x_0) + f_2(y) - f_2(y_0) | \\ &\leq | f_1(x) - f_1(x_0) | + | f_2(y) - f_2(y_0) | \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Now, let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_\phi : V \to V$ be a mapping defined by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_\phi \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+by^{\frac{1}{3}} \\ cx^3+dy \end{smallmatrix}\right)$ and let $g(x, y) = ax + by^{\frac{1}{3}}$ and $h(x, y) = cx^3 + dy$. But the functions k_1, k_2, k_3 and k_4, defined by $k_1(x) = ax, k_2(y) = by^{\frac{1}{3}}, k_3(x) = cx^3$ and $k_4(y) = dy$, are continuous on $\mathbf{R}$. Hence, by Lemma 4.3-2, g and h are continuous on $\mathbf{R}^2 (= V)$.

Next, define $\theta : V \to V$ by

$$\theta(x, y) = (g(x, y), h(x, y)).$$

Let $\varepsilon > 0$. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$| g(x, y) - g(x_0, y_0) | < \frac{\varepsilon}{\sqrt{2}} \quad \text{if} \quad \| (x, y) - (x_0, y_0) \| < \delta,$$

and

$$| h(x, y) - h(x_0, y_0) | < \frac{\varepsilon}{\sqrt{2}} \quad \text{if} \quad \| (x, y) - (x_0, y_0) \| < \delta_2.$$

Let $\delta = min\{\delta_1, \delta_2\}$. If

$$\| (x, y) - (x_0, y_0) \| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

then

$$\| (x, y) - (x_0, y_0) \| < \delta_1 \quad \text{and} \quad \| (x, y) - (x_0, y_0) \| < \delta_2.$$

Therefore

$$\begin{aligned}
\| \theta(x,y) - \theta(x_0,y_0) \| &= \| (g(x,y),h(x,y)) - (g(x_0,y_0),h(x_0,y_0)) \| \\
&= \| (g(x,y) - g(x_0,y_0), h(x,y) - h(x_0,y_0)) \| \\
&= \sqrt{(g(x,y) - g(x_0,y_0))^2 + (h(x,y) - h(x_0,y_0))^2} \\
&< \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} \\
&= \sqrt{\frac{2\varepsilon^2}{2}} \\
&= \varepsilon.
\end{aligned}$$

Consequently $\theta(= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_\phi)$ is continuous on V.

Furthermore, since the sum and the product of continuous functions are continuous, each element of T is continuous on V. However,

$$\lim_{y \to 1} e(1,y) = (0,0) \neq (1,1) = e(1,1).$$

Therefore e is discontinuous at $(1,1)$. Consequently $e \notin T$.

The next note is a summary of the properties of T proved in this section.

Note 4.3-3. (a) T is a near ring with identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_\phi$.

(b) T is 2-primitive on the T-module V.

(c) T is not a ring.

(d) $T \subset N$.

The following theorem is a well-known result by Betsch ([3], Theorem 3.35).

Theorem 4.3-4. *Let R be a near ring with identity which is 2-primitive on the R-module G. Let $C := End_R G$, $D := Aut_R G$. Then $R \subseteq M_D(G)$, $C = D^0$, D is fix point free and either R is a primitive ring on the faithful simple R-module G or R is not a ring and is a dense sub near ring of $M_D(G)$.*

Therefore, by Note 4.3-3, T is a proper dense sub near ring of N.

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